Flow equation, conformal symmetry and AdS geometry

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Abstract

We argue that the Anti-de-Sitter (AdS) geometry in $d+1$ dimensions naturally emerges from an arbitrary conformal field theory in d dimensions using the free flow equation. We first show that an induced metric defined from the flowed field generally corresponds to the quantum information metric, called the Bures or Helstrom metric, if the flowed field is normalized appropriately. We next verify that the induced metric computed explicitly with the free flow equation always becomes the AdS metric when the theory is conformal. We finally prove that the conformal symmetry in d dimensions converts to the AdS isometry in $d+1$ dimensions after d dimensional quantum averaging. This guarantees the emergence of AdS geometry without explicit calculation.

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1 Introduction

The AdS/CFT (or Gravity/Gauge theory) correspondence [1] is a promising tool to crack a hard problem in strongly coupled gauge theories (see [2, 3, 4] for some reviews), but is still mysterious even after many pieces of evidence and application appeared after the first proposal. Although the correspondence may be a manifestation of the closed string/open string duality, an alternative understanding might exist due to its holographic property.

One important mystery is the precise mechanism how the AdS radial direction emerges from conformal field theory (CFT). It may be a common sense that the AdS radial direction is emergent as a renormalization scale of dual CFT as both behave similarly under the dilatational symmetry [1], and this viewpoint worked out for renormalization group flows triggered by relevant deformations [5, 6]. However a direct approach to search the Wilsonian cutoff corresponding to the sharp cutoff on AdS radial direction [7] is still far from a clear answer, since the ordinary Wilsonian renormalization gives rise to non-local interaction in the bulk interpretation.[1] (See also [9].)

While several approaches were developed to *grab the tail* of AdS radial direction and construct bulk dynamics from CFT [10, 11, 12, 13] (see also [14, 15]), one of the present authors, together with his collaborators, has proposed an alternative method to define a geometry from a quantum field theory and explicitly calculated the metric from several quantum field theories [16, 17, 18]. In Ref. [16], the method was proposed and applied to the O(n) non-linear sigma model in 2-dimensions, and the 3 dimensional metric in the large n limit was shown to describe an AdS space in the massless limit. In Ref. [17], it was shown in the large n limit that the induced metric describes an AdS$_{d+1}$ space with $d \geq 3$ in the UV limit and with $d \geq 1$ in the IR limit, if the method is applied to the massive O(n) φ^4 model and an appropriate normalization is introduced for the flowed field. In Ref. [18], the large n expansion was performed for the massless O(n) φ^4 model in 3 dimensions. While the induced metric describes the AdS$_4$ space at the leading order, the next leading order corrections make the space asymptotically AdS only in UV and IR limits with different radii.

By observing that the induced metric always gives the AdS metric when the theory is conformal, it is natural to expect that symmetry plays a key role behind these results. In this paper, we investigate a direct relation between CFT and the AdS metric in this framework. The goal of this paper is to generalize the previous results to an arbitrary conformal field theory incorporating symmetry argument.

The rest of this paper is organized as follows. In Sec. 2, after brief explanation of the proposal of Ref. [16], we first show that our metric corresponds to the information metric stressing importance of the field normalization introduced in Ref. [17]. In Sec. 3, we explicitly derive the AdS metric directly from the CFT. We then prove that the induced metric in $d+1$ dimension possesses the isometry of the AdS space as a consequence of the conformal symmetry in d dimensions. Sec. 4 is devoted to summary and discussion.

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[1]This situation may drastically change in the CFT$_2$/AdS$_3$ correspondence. A clear dictionary was proposed in [8].

2 Gradient flow and information metric

In this section, we briefly review the proposal in Ref. [16, 17] to define a $d+1$ dimensional induced metric from a d dimensional quantum field theory. We also show that the metric defined in this way with appropriate normalization can be interpreted as a quantum information metric, called the Bures or Helstrom metric.

2.1 Gradient flow and induced metric

We consider an n real component scalar field $\varphi(x)$ in d dimensions, whose quantum dynamics is controlled by the action functional $S(\varphi)$. The flowed field ϕ is defined from φ through the flow equation as

$$\frac{1}{R^2}\frac{\partial\phi^a(x;t)}{\partial t} = -\left.\frac{\delta S_f(\varphi)}{\delta\varphi^a(x)}\right|_{\varphi(x)\to\phi(x;t)}, \qquad \phi(x;0)=\varphi(x), \qquad (2.1)$$

where R is a constant with the length dimension, the flow time t is dimensionless, x is the d dimensional coordinate system, $a=1,2,\cdots,n$ labels a component of the scalar field, and $S_f(\varphi)$ is an appropriate action for φ, which is not necessarily related to the original action $S(\varphi)$ in general. Particularly when they coincide, the flow is called the gradient flow [19, 20, 21, 22]. In the case of the free flow (*i.e.* S_f is the free action), the flow equation becomes the heat equation. Thus the flow equation defines a procedure to smear the original field φ into a smeared field ϕ, correlation functions of which are all finite at $t > 0$.

A $d+1$ dimension metric operator is given by

$$\hat{g}_{MN}(x;t) := R^2 \sum_{a=1}^{n} \frac{\partial\sigma^a(x;t)}{\partial z^M}\frac{\partial\sigma^a(x;t)}{\partial z^N}, \qquad (2.2)$$

where $z^M = (x^\mu, \tau)$ with $\tau = \sqrt{2dt}R$, which is regarded as the $d+1$ dimensional coordinates after d dimensional quantum averaging, and $\sigma^a(x;t)$ is the normalized flowed field defined as

$$\sigma^a(x;t) := \frac{\phi^a(x;t)}{\sqrt{\langle\sum_{a=1}^{n}\phi^a(x;t)^2\rangle_S}}. \qquad (2.3)$$

Here $\langle O(\varphi)\rangle_S$ denotes the quantum average with the d dimensional action S, given by

$$\langle O(\varphi)\rangle_S := \frac{1}{Z}\int \mathcal{D}\varphi\, O(\varphi)\, e^{-S(\varphi)}, \quad Z = \int \mathcal{D}\varphi\, e^{-S(\varphi)}. \qquad (2.4)$$

Since $\sigma^a(x;t)$ always satisfies $\sum_{a=1}^{n}\langle\sigma^a(x;t)\sigma^a(x;t)\rangle_S = 1$, we call this normalization condition the non-linear sigma model (NLSM) normalization, whose importance becomes clear in the next subsection. The vacuum expectation value of the metric is denoted as $g_{MN}(z) := \langle\hat{g}_{MN}(x;t)\rangle_S$. See Ref. [16, 17, 18] for more details incorporating the large n limit.

2.2 Information metric

In this subsection, we show that $g_{MN}(z)$ defined in the previous subsection is equivalent to a quantum information metric, called the Bures (or Helstrom) metric. The NLSM normalization is important to show this.

The Bures metric for the density matrix is defined from the infinitesimal distance between two density matrices ρ and $\rho + d\rho$ as

$$D(\rho, \rho + d\rho)^2 = \frac{1}{2}\mathrm{tr}(d\rho\, G), \tag{2.5}$$

where G is the Hermitian 1-form operator implicitly given by $\rho\, G + G\, \rho = d\rho$. In particular for the density matrix ρ of a pure state, the Hermitian operator is determined as $G = d\rho$ since $\rho^2 = \rho$.

In our case, the density matrix for a state localized at $z = (x, \tau)$ is given by

$$\rho_z = |\sigma(x;t)\rangle\langle\sigma(x;t)|, \quad |\sigma(x;t)\rangle := \frac{1}{\sqrt{n}}\sum_{a=1}^{n}|\sigma^a(x;t)\rangle, \tag{2.6}$$

where the inner product of the state is defined as[2]

$$\langle\sigma(x;t)|\sigma(w;s)\rangle := \sum_{a,b=1}^{n}\langle\sigma^a(x;t)\sigma^b(w;s)\rangle_S = \sum_{a=1}^{n}\langle\sigma^a(x;t)\sigma^a(w;s)\rangle_S. \tag{2.7}$$

Notice that the NLSM normalization guarantees $\mathrm{tr}\,\rho_z = 1$. We thus obtain

$$R^2 D(\rho_z, (\rho + d\rho)_z)^2 = \frac{R^2}{2}\mathrm{tr}(d\rho_z d\rho_z) = g_{MN}(z)dz^M dz^N, \tag{2.8}$$

where we used

$$\sum_{a=1}^{n}\langle\sigma^a(x;t)\partial_M\sigma^a(x;t)\rangle_S = 0, \quad \sum_{a=1}^{n}\langle\sigma^a(x;t)\partial_M\partial_N\sigma^a(x;t)\rangle_S = -\sum_{a=1}^{n}\langle\partial_M\sigma^a(x;t)\partial_N\sigma^a(x;t)\rangle_S.$$

Our metric $g_{MN}(z)$ defines a distance in the space of the density matrices made of the pure states $|\sigma(x;t)\rangle$ (in unit of R).

3 AdS isometry from conformal symmetry

In this section, we directly relate arbitrary conformal field theory in the flat d dimensional space-time[3] with the AdS metric in $d + 1$ dimensions. We here assume that the CFT contains a real scalar primary operator $\varphi(x)$ of a general conformal dimension Δ without specifying any concrete models for CFT.

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[2]Thanks to the $O(n)$ symmetry, $\langle\sigma^a(x;t)\sigma^b(w;s)\rangle = \delta^{ab}\langle\sigma^a(x;t)\sigma^a(w;s)\rangle$.

[3] The argument and calculation below hold just by changing the signature suitably when we consider the Euclidean flat space.

3.1 AdS metric from CFT

We start with the free massless flow equation

$$\frac{1}{R^2}\frac{\partial\phi(x;t)}{\partial t} = \partial^2\phi(x;t), \tag{3.1}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. This is easily solved as

$$\phi(x;t) = e^{R^2 t\partial^2}\varphi(x). \tag{3.2}$$

The two-point function of ϕ is evaluated as

$$G_0(x;t|y;s) := \langle\phi(x;t)\phi(y;s)\rangle_{\mathrm{CFT}} = e^{R^2(t\partial_x^2+s\partial_y^2)}\langle\varphi(x)\varphi(y)\rangle_{\mathrm{CFT}}, \tag{3.3}$$

where we use he subscript CFT instead of S, since the action is not specified. The Poincare invariance and the scale transformation of φ with $\varphi(\lambda x) = \lambda^{-\Delta}\varphi(x)$ lead to

$$G_0(x;t|y;s) = \frac{1}{R^{2\Delta}(t+s)^\Delta}F_0\left(\frac{(x-y)^2}{R^2(t+s)}\right). \tag{3.4}$$

Furthermore, the flow equation (3.1) implies

$$\frac{\partial}{\partial t}G_0(x;t|y;s) = R^2\partial_x^2 G_0(x;t|y;s), \tag{3.5}$$

which leads to $F_0(0) = -2dF_0'(0)$. Thus the two-point function of the normalized field σ is

$$G(x;t|y;s) := \langle\sigma(x;t)\sigma(y;s)\rangle_{\mathrm{CFT}} = \left(\frac{2\sqrt{ts}}{t+s}\right)^\Delta F\left(\frac{(x-y)^2}{R^2(t+s)}\right), \tag{3.6}$$

where $F(x) \equiv F_0(x)/F_0(0)$.[4] Hence $F(0) = 1$ and $2dF'(0) = -\Delta$. Taking the $d+1$ dimensional coordinates as $z = (x, \tau = \sqrt{2dt}R)$, the vacuum expectation value of the induced metric $\hat{g}_{MN}$ is

$$g_{MN}(z) := \langle\hat{g}_{MN}(x;t)\rangle_{\mathrm{CFT}} = R^2\langle\partial_M\sigma(x;t)\partial_N\sigma(x;t)\rangle_{\mathrm{CFT}}. \tag{3.8}$$

This is computed as

$$g_{\mu\nu}(z) = \eta_{\mu\nu}\frac{R^2\Delta}{\tau^2}, \quad g_{\mu\tau}(z) = g_{\tau\nu}(z) = 0, \quad g_{\tau\tau}(z) = \frac{R^2\Delta}{\tau^2}, \tag{3.9}$$

which is nothing but the AdS metric with R its radius.

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[4]Explicitly, this is computed as

$$F(x) = \frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(d/2-\Delta)}\int_0^1 dv\, v^{\Delta-1}(1-v)^{d/2-\Delta-1}e^{-xv/4}. \tag{3.7}$$

3.2 Isometry from conformal transformation

In the previous subsection, we have explicitly calculated the induced metric g_{MN} from CFT, and have shown that it becomes the AdS metric in the Poincare patch. In this subsection, we argue the relation between the AdS metric and the CFT only from the symmetry. Namely we show that the induced metric $g_{MN}(z) = \langle \hat{g}_{MN}(x;t) \rangle_{\mathrm{CFT}}$ possesses the isometry of the AdS space. This is necessary and sufficient since AdS is a maximally symmetric space, so that the metric is completely fixed by the isometry group $SO(2,d)$ up to an overall constant.

We first relate the conformal symmetry to the isometry of AdS. The infinitesimal conformal transformation and the response of the primary scalar operator are given by

$$\delta x^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu(b_\nu x^\nu), \tag{3.10}$$

$$\delta^{\mathrm{conf}}\varphi(x) = -\delta x^\mu \partial_\mu \varphi(x) - \frac{\Delta}{d}(\partial_\mu \delta x^\mu)\varphi(x). \tag{3.11}$$

Here $a^\mu, \omega^{\mu\nu}, \lambda, b^\mu$ are parameters of the transformation. Since the infinitesimal conformal transformation is quadratic in the coordinate x, the normalized field σ, given in terms of φ by

$$\sigma(x;t) = \frac{\left(\sqrt{2t}\right)^\Delta}{\sqrt{F_0(0)}} e^{R^2 t \partial^2} \varphi(x), \tag{3.12}$$

is transformed as

$$\delta^{\mathrm{conf}}\sigma(x;t) = -\left\{ R^2 t(\partial^2 \delta x^\mu) + 2R^4 t^2(\partial^\nu \partial^\rho \delta x^\mu)\partial_\nu \partial_\rho + 2R^2 t(\partial^\nu \delta x^\mu)\partial_\nu + \delta x^\mu \right\} \partial_\mu \sigma(x;t)$$
$$- \frac{\Delta}{d} \left\{ 2R^2 t(\partial^\nu \partial_\mu \delta x^\mu)\partial_\nu + (\partial_\mu \delta x^\mu) \right\} \sigma(x;t). \tag{3.13}$$

Plugging (3.11) into (3.13) we obtain

$$\delta^{\mathrm{conf}}\sigma(x;t) = -\{2\lambda - 4(b_\mu x^\mu)\}t\partial_t \sigma(x;t) - \{\delta x^\mu + 2R^2 t(d - 2 - \Delta)b^\mu\}\partial_\mu \sigma(x;t)$$
$$+ 4R^2 t^2 b^\mu \partial_\mu \partial_t \sigma(x;t), \tag{3.14}$$

where we have used $\partial_t \sigma(x;t) = (\frac{\Delta}{2t} + R^2 \partial^2)\sigma(x;t)$.

We then rewrite eq. (3.14) as

$$\delta^{\mathrm{conf}}\sigma(x;t) = \delta^{\mathrm{diff}}\sigma(x;t) + \delta^{\mathrm{extra}}\sigma(x;t) \tag{3.15}$$

where

$$\delta^{\mathrm{diff}}\sigma(x;t) = -(\bar{\delta}t\partial_t + \bar{\delta}x^\mu \partial_\mu)\sigma(x;t), \quad \delta^{\mathrm{extra}}\sigma(x;t) = 4R^2 t^2 b^\nu \partial_\nu(\partial_t + \frac{\Delta + 2}{2t})\sigma(x;t) \tag{3.16}$$

with $\bar{\delta}x^\mu = \delta x^\mu + 2dR^2 tb^\mu, \bar{\delta}t = (2\lambda - 4(b_\mu x^\mu))t$. By setting $\tau^2 = 2dR^2 t$, the transformation δ^{diff} can be rewritten as

$$\bar{\delta}x^\mu = \delta x^\mu + \tau^2 b^\mu, \quad \bar{\delta}\tau = (\lambda - 2(b_\mu x^\mu))\tau. \tag{3.17}$$

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This is nothing but the isometry transformation of the AdS space, whose metric is given by $ds_{AdS}^2 \propto \dfrac{d\tau^2 + dx^{\mu 2}}{\tau^2}$.

The conformal transformation of the induced metric operator is computed as

$$\delta^{\mathrm{conf}} \hat{g}_{MN}(x;t) = \delta^{\mathrm{diff}} \hat{g}_{MN}(x;t) + R^2 \lim_{(y;s)\to(x;t)} \frac{\partial}{\partial z^M} \frac{\partial}{\partial w^N} \left\{ \delta^{\mathrm{extra}} \sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\mathrm{extra}}\sigma(y;s) \right\}.$$

The first term is

$$\delta^{\mathrm{diff}} \hat{g}_{MN}(x;t) = -\bar{\delta} z^K \partial_K \hat{g}_{MN}(x;t) - \partial_M \bar{\delta} z^K \hat{g}_{KN}(x;t) - \partial_N \bar{\delta} z^K \hat{g}_{MK}(x;t), \qquad (3.18)$$

which is nothing but the diffeomorphism of the metric tensor in the $d+1$ dimensions.

Thus our task is to show that the second term in eq. (3.18) vanishes in the vacuum expectation value. Since

$$\left\langle \delta^{\mathrm{extra}} \sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\mathrm{extra}}\sigma(y;s) \right\rangle$$
$$= -8 \frac{(\sqrt{4ts})^\Delta}{(t+s)^{\Delta+2}} (t-s) b_\mu (x-y)^\mu \frac{(x-y)^2}{R^2} F'' \left(\frac{(x-y)^2}{R^2(t+s)} \right) \qquad (3.19)$$

it is easy to see[5]

$$\lim_{(y;s)\to(x;t)} \frac{\partial}{\partial z^M} \frac{\partial}{\partial w^N} \left\langle \delta^{\mathrm{extra}} \sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\mathrm{extra}}\sigma(y;s) \right\rangle = 0. \qquad (3.20)$$

Therefore we obtain
$$\left\langle \delta^{\mathrm{conf}} \hat{g}_{MN}(x;t) \right\rangle = \left\langle \delta^{\mathrm{diff}} \hat{g}_{MN}(x;t) \right\rangle. \qquad (3.21)$$

Since the conformal invariance of the two point function of the primary scalar operator implies that $\left\langle \delta^{\mathrm{conf}} \hat{g}_{MN}(x;t) \right\rangle = 0$, it follows that $\left\langle \delta^{\mathrm{diff}} \hat{g}_{MN}(x;t) \right\rangle = 0$. This means that the induced metric $g_{MN}(z)$ satisfies the Killing equation of the AdS space as

$$\delta^{\mathrm{diff}} g_{MN}(z) = -\bar{\delta} z^K \partial_K g_{MN}(z) - \partial_M \bar{\delta} z^K g_{KN}(z) - \partial_N \bar{\delta} z^K g_{MK}(z) = 0, \qquad (3.22)$$

which implies that the induced metric must be the AdS metric up to an overall constant. This completes the proof of our claim.

4 Discussion

In this paper an induced geometry by a flow equation from a quantum field theory was investigated. The induced metric was shown to appear as a quantum information metric, which measures a distance in the space of the pure states constructed by scalar fields in a general quantum field theory. In a conformally symmetric situation, it was shown that the

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[5]The quantum average and the derivative commute since all correlation functions of σ are finite as long as the flow time is non-zero.

induced metric matches the AdS one when the flow equation is free and massless[6]. This agreement was confirmed only by using symmetry without any explicit computations of the metric.

So far any relation between the induced metric formalism presented in this paper and other approaches to see dual geometry mentioned in the introduction is not known. It may be reasonable to think that there is no relation between them since, for example, the procedure to renormalize fields in quantum field theory and that to smear operators are generally independent. Still we expect that the results and technique developed in this paper, particularly the symmetry argument, will become useful to study the AdS geometry from CFT by another method.

In this paper we assume that the background of quantum field theory or conformal field theory is flat. It would be interesting to extend the presented calculation to curved backgrounds. In particular it would be curious to check whether the induced metric from CFT on a curved space-time is still of the AdS form in a different coordinate system from the Poincare patch.

A challenging but important issue is whether this formalism encodes the gravitational dynamics or not. The first step toward this goal may be to see how a linearized Einstein gravity is encoded in this formulation, as shown in a different method to derive dual bulk dynamics by using the entanglement entropy [23, 24]. For this analysis it will be necessary to specify a concrete model to test the proposal such as an $O(n)$ sigma model, since bulk dynamics is dependent on each CFT. A virtue of this formulation is that observables in this formalism are correlation functions of scalar fields in quantum field theory, which admit analytic computation by the ordinary technique of the $1/n$ expansion [18], so that one can proceed by checking one's guesswork explicitly by hand.

We hope to report on resolutions on these issues in the near future.

Acknowledgement

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