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A COMPARATIVE STUDY OF UNCERTAINTY PROPAGATION METHODS FOR BLACK-BOX TYPE FUNCTIONS

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ABSTRACT

It is an important step in deign under uncertainty to select an appropriate uncertainty propagation (UP) method considering the characteristics of the engineering systems at hand, the required level of UP associated with the probabilistic design scenario, and the required accuracy and efficiency levels. Many uncertainty propagation methods have been developed in various fields, however, there is a lack of good understanding of their relative merits. In this paper, a comparative study on the performances of several UP methods, including a few recent methods that have received growing attention, is performed. The full factorial numerical integration (FFNI), the univariate dimension reduction method (UDR), and the polynomial chaos expansion (PCE) are implemented and applied to several test problems with different settings of the performance nonlinearity, distribution types of input random variables, and the magnitude of input uncertainty. The performances of those methods are compared in moment estimation, tail probability calculation, and the probability density function (PDF) construction. It is found that the FFNI with the moment matching quadrature rule shows good accuracy but the computational cost becomes prohibitive as the number of input random variables increases. The accuracy and efficiency of the UDR method for moment estimations appear to be superior when there is no significant interaction effect in the performance function. Both FFNI and UDR are very robust against the non-normality of input variables. The PCE is implemented in combination with FFNI for coefficients estimation. The PCE method is shown to be a useful approach when a complete PDF description is desired. Inverse Rosenblatt transformation is used to treat non-normal inputs of PCE, however, it is shown that the transformation may result in the degradation of accuracy of PCE. It is also shown that in black-box type of system the performance and convergence of PCE highly depend on the method adopted to estimate its coefficients.

KEYWORDS

Uncertainty propagation, Full factorial numerical integration, Dimension reduction method, Polynomial chaos expansion, Comparative study

1. INTRODUCTION

One of the key components of uncertainty analysis is the quantification of uncertainties in the system output performances propagated from uncertain inputs, named as uncertainty propagation (UP). A lot of efforts have been made to develop methods of uncertainty propagation in various fields such as structural reliability (Madsen et al. 2006, Christensen 1982, Kiureghian 1998), stochastic mechanics (Ghanem and Spanos 1991, Liu et al. 1986), quality engineering (Evans 1972, Taguchi 1978, D'Errico and Zaino 1988, Seo and Kwak 2002) and considerably many methods are now available. Since the methods were developed with different backgrounds and philosophies, it is necessary to examine their applicability and relative efficiency in the context of engineering design. In design under uncertainty, the role that the UP is expected to play varies based on different design scenarios. For instance, in robust design (Du and Chen 2000, McAllister and Simpson 2003), the interest of UP is to evaluate the low-order moments (mean and variance) of a performance. In reliability-based design (Lee and Kwak 1987-88, Wu 1994, Youn et al. 2003, Du et al. 2004), the interest is on assessing the performance reliability. The complete distribution of a performance is needed in utility optimization (Hazelrigg 1998), where the probability distribution needs to be integrated with designer's utility function to maximize the expected utility of a product. Our goal in this work is to conduct an in-depth examination of several widely used UP techniques, some are newly developed in literature, to understand the characteristics and limitations of these methods, and to compare their performance using illustrative examples.

The methods for uncertainty propagation can be classified into five categories as follows. The first category is the

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simulation based methods like Monte Carlo simulation (MCS) (Madsen et al. 2006, Christensen 1982, Kiureghian 1998), importance sampling (Melchers 1989, Engelund and Rackwitz 1993), adaptive sampling (Bucher 1988). The second category is the local expansion based methods like Taylor series method or perturbation method (Madsen et al. 2006, Christensen 1982, Kiureghian 1998, Ghanem and Spanos 1991). Methods in this category are weak against the large variability of inputs and nonlinearity of performance functions. The third category is the most probable point (MPP) based methods (Hasofer and Lind 1974, Fiessler et al. 1979). First order reliability method (FORM) and second order reliability methods (SORM) are two popular methods in this category. The fourth category is the functional expansion based methods. The Neumann expansion and the polynomial chaos expansion can be classified into this group. The polynomial chaos expansion (PCE) (Ghanem and Spanos 1991, Xiu and Karniadakis 2003) is a mean square convergent series of orthogonal polynomials of standard random variables. In recent years, the PCE approach has been gaining more attention in uncertainty representations, stochastic mechanics, solution of stochastic differential equations and so on. The last category is the numerical integration based methods (Evans 1972, Seo and Kwak 2002, Lee and Kwak 2006). The statistical moments are first calculated by direct numerical integration and then the probability density or the tail region probability is approximated using the empirical distribution systems (Johnson et al. 1994) based on the calculated moments. This approach has been implemented together with the technique of design of experiments and the procedure using the product quadrature rule with Gaussian type integration and Pearson system for reliability analysis was named as the full factorial moment method (FFMM) in Seo and Kwak (2002) and Lee and Kwak (2006). For clarity, in this paper, we denote the moment calculation with full factorial set of evaluation points as full factorial numerical integration (FFNI). A newly developed method in this last category of numerical integration based method is the so called dimension reduction (DR) method (Rhaman and Xu 2004, Xu and Rhaman 2004). The DR method approximates a multi-dimensional moment integral by multiple reduced-dimensional integrals based on additive decomposition of performance function.

When selecting a method for UP various aspects should be considered such as the required level of uncertainty quantification, accuracy or confidence level, as well as the computational cost or efficiency. Since the performance of the above mentioned methods are affected by the problem settings such as the type of input distribution, the nonlinearity of performance function, the number of input random variables, and the required resolution, it would be beneficial to develop guidelines to choose an appropriate method which fits for a specific situation. For this, comparative studies among the various methods are important. So far, comparative studies among the numerical integration based methods (e.g., the dimension reduction method) and the functional expansion based methods, especially the PCE method have not been reported in literature. In this paper, we present the results of our comparative study on the performance of a few relatively recent methods, such as the Univariate Dimension Reduction (UDR) and the PCE, as well as a few conventional approaches, such as the FFNI and the MPP based approaches. In our studies, the results from MCS are used as a reference. Since in most of the engineering design problems, the analysis models are provided in the form of black-boxes rather than explicit functions or governing equations, the black-box type problems are the target of our study. Even though explicit functions are provided for our case studies, they are treated as black-box type functions with different degrees of nonlinearity and different distribution types of input variables.

In Section 2, introductions to FFNI, UDR and PCE are provided with emphasis on the unique aspects of each method. Test results with several problems are given in Section 3. In Section 4, a brief summary and discussions about the results are provided.

2. SUMMARY OF METHODS

Throughout this paper, a black-box type performance function is denoted as

$$y = g(\mathbf{x}) \tag{1}$$

where $\mathbf{x}$ is the n dimensional vector of random variables with joint probability density function (PDF) $f(\mathbf{x})$.

2.1 Full Factorial Numerical Integration (FFNI)

With this approach, the statistical moments are calculated through the direct numerical integration. Once the first four moments are calculated, the complete distribution function or the probability of failure can be estimated using the empirical distribution systems (Johnson et al. 1994). Some details are provided as follows.

The *m*-node Gaussian type integration rule for statistical moments can be written as

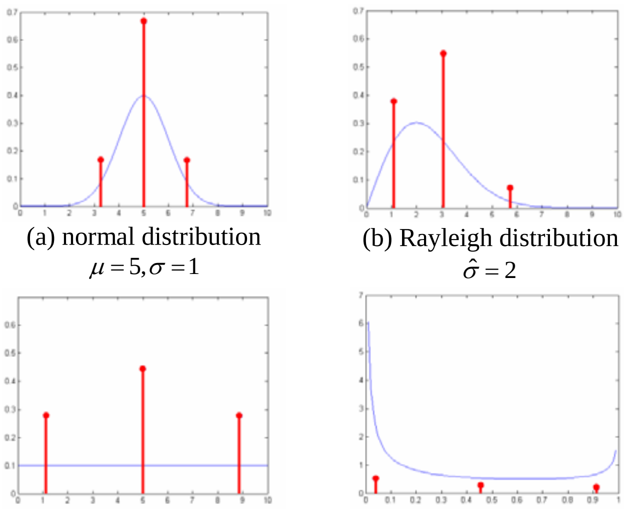
$$E\left[g^k\right] = \int_\Omega \left\{g(x)\right\}^k f(x)dx \approx \sum_{i=1}^{m} w_i \left[g(\mu + \alpha_i \sigma)\right]^k \tag{2}$$

where α_i, w_i are the location parameter and weight at the i-th quadrature point. The optimal locations of the quadrature points and the corresponding weights can be calculated using the moment-matching equations as follows:

$$M_k = \int_\Omega (x-\mu)^k f(x)dx = \sum_{i=1}^{m} w_i (\alpha_i \sigma)^k \qquad k = 0,...,2m-1 \tag{3}$$

where M_k is the k-th central moment of random variable x which should be provided based on the PDF of x. This nonlinear system of equations can be solved with numerical methods to find the unique $\{\alpha_1,\cdots,\alpha_m,w_1,\cdots,w_m\}$. However, when m becomes large (e.g. > 7), it is not a simple task to solve the nonlinear system of equations. Fig. 1 is the example of evaluation points and weights for some well-known distributions when the 3-node rule is applied ($m=3$).

When x follows the normal, uniform and exponential distribution, α_i's and w_i's can be directly derived from the Gauss-Hermite, Gauss-Legendre, and Gauss-Laguerre quadrature formula respectively since the weighting functions of those quadrature rules have the same form with the PDF of the above three distributions. When there are n random variables in the system, the moment integral becomes an n dimensional multiple integral based on the product quadrature rule (Evans 1972) and can be calculated as follows:



(a) normal distribution $\mu = 5, \sigma = 1$

(b) Rayleigh distribution $\hat{\sigma} = 2$

(c) uniform distribution $0 \le x \le 10$

(d) beta distribution $\eta = 0.3, \gamma = 0.6$

Figure 1. PDF and 3-node quadrature rule for four different distributions. The notations for parameters are from Hahn and Shapiro (1967).

$$
\begin{aligned}
E\left\{g^k\right\} &= \int_{\Omega_1} \cdots \int_{\Omega_n} \left[g\left(x_1, \cdots, x_n\right)\right]^k f\left(x_1, \cdots, x_n\right) dx_1 \cdots dx_n \\
&\cong \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_n=1}^{m} w_{j_n} \left[g(\mu_1 + \alpha_{j_1}\sigma_1, \cdots, \mu_n + \alpha_{j_n}\sigma_n)\right]^k
\end{aligned} \tag{4}
$$

The sampling becomes a m^n full factorial design in the DOE point of view. The integration order of this method is $(2m-1)$ and the number of function evaluations required is m^n.

Once the four statistical moments are obtained, the probability of failure can be evaluated with the aids of the empirical distribution systems such as the Pearson system, the Johnson system, and the Gram-Charlier series. Pearson system approximates the PDF $f(x)$ as a solution of the differential equation following,

$$
\frac{1}{f(x)}\frac{df(x)}{dx} = -\frac{\bar{x} + a}{c_0 + c_1\bar{x} + c_2\bar{x}^2} \tag{5}
$$

where $\bar{x} = x - \mu$ and, a, c_0, c_1 and c_2 are coefficients calculated from the first four central moments of random variable x whose PDF is to be approximated.

2.2 Univariate Dimension Reduction (UDR) Method

Similar to the FFNI, the univariate dimension reduction (UDR) method (Rhaman and Xu 2004, Xu and Rhaman 2004) belongs to the category of numerical integration based methods. This method calculates the statistical moments of $g(\mathbf{x})$ with a univariate decomposition to improve the efficiency of numerical integration. The performance function $g(\mathbf{x})$ is approximated by a sum of univariate functions which depend on only one random variable with the other variables fixed to their mean values. If we denote these univariate functions as g_i then the approximation can be written as:

$$
\begin{aligned}
g(\mathbf{x}) \approx \hat{g}(\mathbf{x}) &= \sum_{i=1}^{n} g\left(\mu_1, \cdots, x_i, \cdots, \mu_n\right) - (n-1)\, g\left(\mu_1, \cdots, \mu_n\right) \\
&= \sum_{i=1}^{n} g_i\left(x_i\right) - (n-1)\, g\left(\boldsymbol{\mu}_{\mathbf{x}}\right)
\end{aligned} \tag{6}
$$

where μ_i denotes the mean value of *i*-th random variable.

It can be shown that the Taylor series expansion of the univariate approximation $\hat{g}(\mathbf{x})$ contains all single variable terms of Taylor series of $g(\mathbf{x})$. This means that the approximation error is contributed by the terms with two or more variables in the expansion of $g(\mathbf{x})$.

The univariate decomposition can be applied to the statistical moment integral, with the condition that the random variables should be independent with each other. When the variables are correlated, they should be transformed into independent variables with Rosenblatt transformation. The k-th moment of $g(\mathbf{x})$ is approximated as follows:

$$
E\left[g^k(\mathbf{x})\right] \approx E\left[\hat{g}^k(\mathbf{x})\right] = E\left[\left\{\sum_{i=1}^{n} g_i\left(x_i\right) - (n-1)\, g\left(\boldsymbol{\mu}_{\mathbf{x}}\right)\right\}^k\right] \tag{7}
$$

This operations can be performed algebraically and the results can be expressed in terms of moments of univariate functions, say $E\left[g_i^l\right]$, $i = 1, \cdots, n$ $\quad l = 0, \cdots, k$. In Rhaman and Xu (2004), a recursive formula for this calculation is proposed.

The moments of g_i's can be calculated using the one dimensional numerical integration based on the moment-based quadrature rule. Schemes introduced in Section 2.1 for sampling can also be used. If we use the quadrature rule with the same number of nodes m for one dimensional integration of all g_i's, then the number of $g(\mathbf{x})$ evaluations required becomes $(m-1)n+1$ at least and $mn+1$ at most (n: number of random variables). Since the output of UDR is also statistical moments of $g(\mathbf{x})$, the empirical distribution system introduced in Section 2.1 needs to be applied to get the full distribution function to calculate the probability of failure.

2.3 Polynomial chaos expansion (PCE)

The PCE method belongs to the category of *functional expansion based methods*. The PCE of a square integrable random variable $u(\theta)$ can be written as (Ghanem and Spanos 1991)

$$
\begin{aligned}
u(\theta) &= a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1\left(\xi_{i_1}(\theta)\right) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2\left(\xi_{i_1}(\theta), \xi_{i_2}(\theta)\right) \\
&+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3\left(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)\right) + \cdots
\end{aligned} \tag{8}
$$

where $\left\{\xi_i(\theta)\right\}_{i=1}^{\infty}$ is a set of standard normal variables, Γ_p is a generic element in the set of multi-dimensional Hermite polynomials of order p, and a_i are the deterministic coefficients. θ is a parameter indicating the quantities involved are random variables defined over a space of random events. We can rewrite Eq. (8) in a simpler form as

$$u(\theta)=\sum_{i=0}^{\infty} b_i \Psi_i(\xi(\theta)) \tag{9}$$

where b_i, $\Psi_i(\xi)$ correspond to $a_{i_1\cdots i_p}$, $\Gamma_p(\xi_{i_1},\cdots,\xi_{i_p})$ respectively. For example, the two dimensional PCE of second order ($p=2$) can be written as

$$\begin{aligned} u(\theta) &= b_0 + b_1\xi_1(\theta) + b_2\xi_2(\theta) + b_3(\xi_1^2(\theta)-1) \\ &+ b_4\xi_1(\theta)\xi_2(\theta) + b_5(\xi_2^2(\theta)-1) \end{aligned} \tag{10}$$

Some notable properties of Hermite polynomials are

$$E[\Psi_i \Psi_j] = E[\Psi_i^2]\delta_{ij} \tag{11}$$

$$E[\Psi_i] = 0 \quad for \;\; i \neq 0 . \tag{12}$$

When there are n random inputs in the system as in Eq. (1), the output responses can be approximated by n dimensional PCE, truncated at some order p. In this case the number of terms in PCE becomes $P+1$ where P is given as

$$P=\sum_{s=1}^{p}\frac{1}{s!}\left\{\prod_{r=0}^{s-1}(n+r)\right\} . \tag{13}$$

Then the PC approximation is written as

$$y \approx y^{(P)} = \sum_{i=0}^{P} b_i \Psi_i(\xi) \tag{14}$$

where ξ is a standard normal random vector of n dimension.

The coefficients of Eq. (14) can be calculated based on the orthogonality (Eq. (11)) of Hermite polynomials. Multiplying $\Psi_j(\xi)$ on both sides of Eq. (14) and taking the expectation, we can obtain b_j as follows:

$$b_j = E[y\Psi_j(\xi)] \Big/ E[\Psi_j^2(\xi)] \tag{15}$$

While the denominator term in Eq. (15) can be evaluated analytically, the expectation in the numerator needs to be evaluated with sampling or numerical integration schemes. Approaches using Latin Hyper cube sampling (Choi et al. 2004), numerical integration (Xiu 2007) have been reported. One issue related with the procedure is how to treat the non-normal random inputs. It is reported that the convergence rate of PCE is not fast when the inputs are non-normal and a generalized PCE (Xiu and Karniadakis 2003) has been proposed to solve this problem with various orthogonal polynomials in Askey scheme. Using the generalized PCE is one way to treat non-normal random inputs, and the other way is to use transformation (Tatang 1995). The comparison between these two approaches, using the generalized PCE and using transformation, was performed by Choi et al. (2004). It is reported that the use of generalized PCE guarantees faster convergence but the use of transformation is easier in implementation.

In our implementation, m^n FFNI with inverse Rosenblatt transformation is used. The detailed procedure of calculating coefficients in Eq. (15) is as follows:

1) Define transformations which map random variable x_i to independent standard normal variable ξ_i as

$$x_i = T_i(\xi_i) = F^{-1}(\Phi(\xi_i)) \;\; i=1,\cdots,n \tag{16}$$

where F denotes the cumulative distribution function (CDF) of the original random variable and Φ is the CDF of the standard normal distribution. It is assumed that x_i are independent with each other.

2) Set up the integration points and weights $\{l_{i\cdot j}, w_{i\cdot j}\}$.

3) The coefficients b_i is calculated as

$$b_i = \left[\sum_{j_1=1}^{m} w_{j_1}\cdots\sum_{j_n=1}^{m} w_{j_n} g\left(T_1(l_{1\cdot j_1}),\cdots,T_n(l_{n\cdot j_n})\right)\Psi_i(l_{1\cdot j_1},\cdots,l_{n\cdot j_n})\right] \Big/ E[\Psi_i^{\,2}] \tag{17}$$

where the expectation in denominator can be evaluated analytically.

The output of PCE is a random variable expressed in terms of standard normal variables and this is a different aspect of PCE from other methods whose outputs are usually measures of uncertainty such as moments. The above procedure of PCE can be thought of as a construction of a non-normal random variable with projection onto the orthogonal basis of random variable space based on the observational data. Once the expansion function is obtained, the moments and probability of failure can be derived if needed. This feature of obtaining directly the PDF function of an output response using the PCE method is very useful when the complete PDF construction is needed, either for evaluating the probabilistic design objective or for further propagation of uncertainty in a chain type of system.

3. COMPARATIVE STUDIES

Four examples are tested to compare the performance of the methods mentioned, under different levels of function nonlinearity and different types of uncertain inputs. The first two examples are chosen to compare the performances of FFNI, UDR, and PCE in moment estimation. Three and five-node (*m* = 3 or 5) quadrature rules are used for one dimensional integration in both FFNI and UDR. In PCE, the expansions up to 4-th order (*p*=4) are tried with coefficients calculated by the procedure described in Section 2.3 with 5^n FFNI. When the PCE method is used, the moments are calculated analytically and the probability of failure is estimated by MCS based on the PCE functions. In the last two problems, the FORM is applied to compare the accuracy and efficiency of probability calculation with the other three methods. The HL-RF algorithm (Hasofer and Lind 1974, Rackwitz and Fiessler 1978) is used to find the MPP.

3.1 Example 1

$$y = x^k \quad k=1,2,\cdots,7 \tag{18}$$

This example is used to compare the accuracy of various methods in moment estimation against the nonlinearity of performance function in an one dimensional case. Two cases are tested, one with an input that follows the normal distribution and the other with the lognormal distribution. Since this problem is one dimensional, the test of UDR method is not included because there is no need for dimension reduction.

3.1.1 Case 1: $x \sim$ normal $(\mu=1, \sigma=0.2)$

In Figs. 2 and 3, the skewness and kurtosis calculation results are depicted with the ratio to the exact values obtained analytically. NI in the legend means the method using numerical integration. Results of MCS with 1,000 k samples are provided together for comparison. Since the input x follows a normal distribution, the Gauss-Hermite quadrature rule with m-node (either 3 or 5) is used for numerical integration and $2m-1$ integration order is expected. It means that when m is 5 the skewness results are exact up to nonlinear order $k=3$ and kurtosis results are exact up to $k=2$. The 5 node numerical integration seems sufficient for calculating the coefficients of PCE up to 4-th order in this case. PCE with coefficients obtained by numerical integration, Eq. (17) is compared with PCE with coefficients calculated analytically in Appendix. As the nonlinearity of $g(x)$ increases, the coefficients become erroneous for the higher order terms in the PCE. The reason why higher order expansion is still more accurate under the bigger coefficient estimation error is that the truncation error is more significant than the numerical error in the coefficient estimation. It is notable that for any order of expansion, $E\left[y^{(P)}\right]$ equals $E[y]$ based on the property in Eq. (12). Due to the orthogonality, the coefficients of low order terms stay the same even if higher order terms are added.

From Figs. 2 and 3, it is noted that for this example, for different orders of nonlinearity k, the accuracy follows the sequence of MCS (highest), 4th order PCE, 5 node NI, 3rd order PCE, 2nd order PCE, and the 3 node NI (lowest). The 4-th order PCE shows slightly better results in skewness and kurtosis calculation than the 5-node numerical integration while both methods use the same amount of samples (5). Comparing the results of different orders of PCE, we can check the amount of truncation error when a lower order of PCE is used. On the other hand, the higher order expansion we choose, the greater the error might become in the coefficient of the expansion function. Further study is necessary on choosing the right order based on the balance between the truncation error and the coefficient estimation error. It is observed that with the increase of the PCE order from 2 to 4, the PDF curve is converged to the Exact curve (Fig. 4).

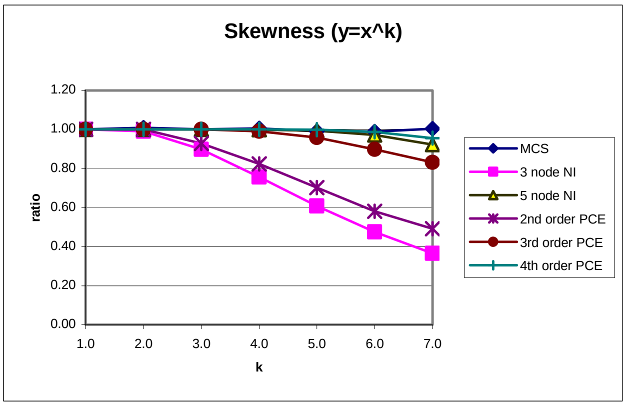


Figure 2. Comparison of results in skewness estimation (ratio= result/exact value)

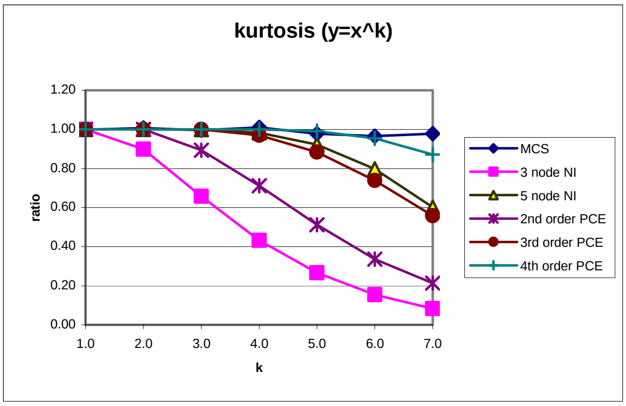


Figure 3. Comparison of results in kurtosis estimation (ratio=result/exact value)

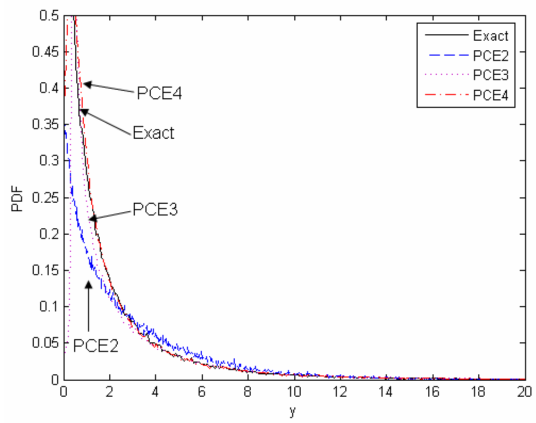


Figure. 4 Probability density function estimated by PCE (k=7)

3.1.2 Case 2: x ~ lognormal $(\mu=1, \sigma=0.2)$

The same problem is tested with an input variable following the lognormal distribution. In numerical integration and MCS, no transformation is involved but for PCE, a transformation to the standard normal variable should be used during the calculation of coefficients. In this example, the mapping between the normal variable and the lognormal variable is available as

$$x=\exp(\hat{\sigma}\xi+\hat{\mu}) \quad \xi \sim N(0,1^2)$$
$$\hat{\mu}=\log\mu_x-0.5\log(\sigma_x^2+1),\ \hat{\sigma}=\sqrt{\log(\sigma_x^2+1)} \qquad (19)$$

where μ_x and σ_x denote the mean and standard deviation of x, respectively.

The results of skewness and kurtosis calculation are summarized in Figs. 4 and 5. In the numerical integration, the evaluation points and weights are obtained by solving the moment-matching equation, Eq. (3). Although the results show bigger error for large k values (order of nonlinearity) compared to the results of case 1, we can see that the numerical integration with the moment matching quadrature rule provides accurate results up to $2m-1$ polynomial order.

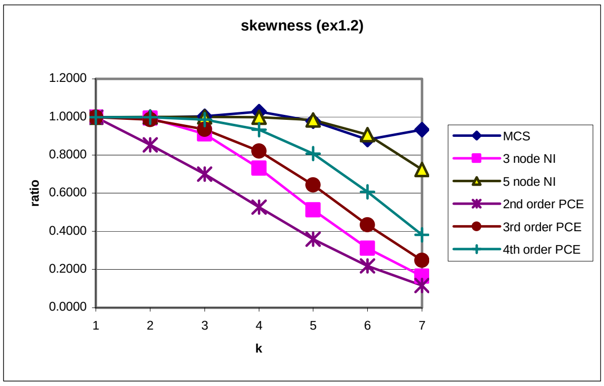


Figure. 5. Comparison of results in skewness estimation (ratio= result/exact value)

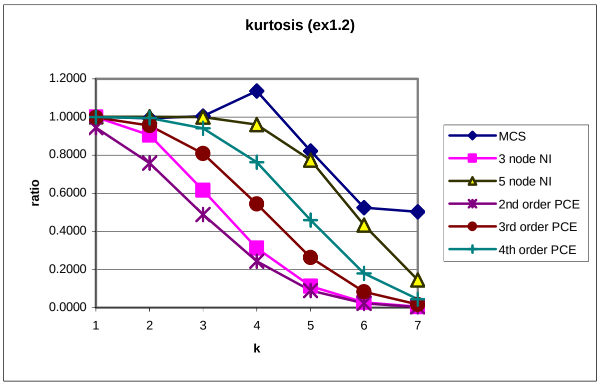


Figure 6. Comparison of results in kurtosis estimation (ratio= result/exact value)

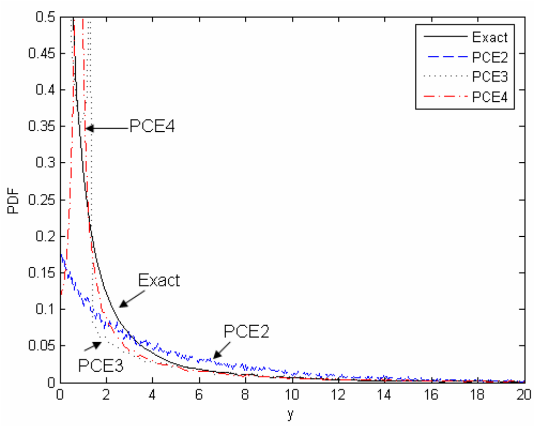


Figure 7. Probability density function estimated by PCE (k=7)

In Case 2, the accuracy follows the sequence of MCS (highest), 5 node NI , 4th order PCE, 3rd order PCE, 3 node NI, and 2nd order PCE (lowest). Contrary to the result of the normal input case, the accuracy of 4-th order PCE is worse than the 5 node numerical integration and that of the 2nd order PCE is worse than the 3 node NI, which implies that the transformation does have a negative effect on the results. The nonlinearity of y is amplified with the transformation in Eq. (19) and the coefficient estimation becomes erroneous when the nonlinearity exceeds the integration order of the quadrature rule. When checking the convergence of PDF obtained by PCE, it is shown that the convergence behavior is not as good as in Case 1 with normal distributions.

3.2 Example 2

The second example is a three dimensional polynomial

$$y = x_1^a x_2^a + 2x_3^4 \qquad (20)$$

where x_i's follow the lognormal distribution with mean 1. Four values of the standard deviation, 0.1, 0.2, 0.3 and 0.4 are tested to examine the effect of variability of inputs on the moment estimation. Also we intend to observe the effect of interactions among variables on the performance of univariate dimension reduction method by trying several values of a in Eq. (20). When calculating the PC coefficients, the transformation in Eq. (19) is used again.

3.2.1 case 1: $a = 2$

The results of standard deviation and skewness calculations are summarized in Figs. 8 and 9. The coefficients of PCE are calculated using the procedure described in Section 2.3 with 5^n FFNI. From the figures it is seen that the results of UDR is almost identical with that from the FFNI, which implies that the univariate decomposition is valid for y, that is, the interaction effects among variables are not significant in this example. From Table 1, we note that the number of function evaluations for UDR is much less than that used for the other methods. The result of PCE is much worse than that from the 5^n FFNI which is due to the non-normality of the input random variables as in case 2 of example 1. Transformation does not always result in the degradation of accuracy, however, we can see that care must be taken when integrating a function with transformed variables. UDR shows an excellent efficiency compared to FFNI. On the hand, the performance of MCS is even worse compared to the 5^n FFNI and 5n UDR.

Table 1 Number of function evaluations in Ex 2

Method	MCS	3^n FFNI	5^n FFNI	3n UDR	5n UDR	4th PCE
Fn calls	1,000 k	27	125	7	13	125

3.2.2 case 2: $a = 3$

By increasing a, we expect that the interaction effect between x_1 and x_2 becomes more significant. The analysis results of the standard deviation and skewness are summarized in Table 2~3. The computational costs are the same as in case 1. Compared to the results of case 1, the results of UDR show considerable discrepancies with those of MCS, and FFNI especially when the standard deviations (σ) of variables are big. This implies that when the interaction effects are strong, the error of using UDR increases.

Table 2 Standard deviation calculation results for Ex 2 (case 2)

σ	MCS	3^nFFNI	5^nFFNI	3n UDR	5nUDR	4th PCE
0.1	1.0001	0.9990	0.9994	0.9879	0.9883	0.9994
0.2	2.6854	2.6723	2.6894	2.5634	2.5811	2.6851
0.3	6.5869	6.4263	6.6237	5.9017	6.1114	6.4627
0.4	17.5815	16.2045	17.6457	14.0807	15.6562	15.5571

Table 3 Skewness calculation results for Ex 2 (case 2)

σ	MCS	3^nFFNI	5^nFFNI	3n UDR	5n UDR	4^{th} PCE
0.1	1.0541	0.9910	1.0530	0.9345	0.9982	1.0540
0.2	2.9137	2.2053	2.8979	2.1221	2.8809	2.7196
0.3	7.2888	4.0926	8.2544	3.8275	8.7001	5.2245
0.4	24.6705	8.2238	31.3577	13.7716	33.7417	7.6342

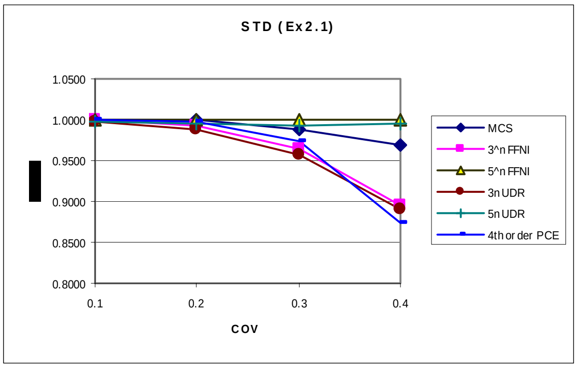


Figure 8. Comparison of results of STD estimation (ratio=result/exact value)

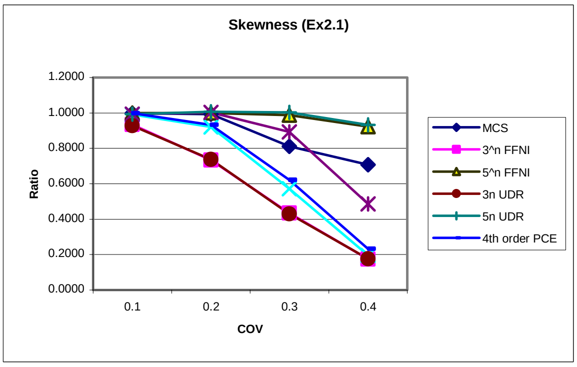


Figure 9. Comparison of results of skewness estimation (ratio=result/exact value)

3.3 Engine piston power loss analysis

This example is from the Piston-ring/Cylinder-liner example used by Kokolaras et al. (2006), Huibin et al. (2006). The performance of interest is the power loss due to the friction between the piston ring and the cylinder liner. Our interest is to find the probability density function over the whole range of power loss and also compute the probability in the tail region. The prediction model for power loss due to friction is a meta-model built by the moving least square (MLS) method with four inputs, ring surface roughness, liner surface roughness, liner Young's modulus, and liner hardness. The first two inputs are treated as random variables following normal distributions with mean 4.0 and 6.1193 respectively and with unit variance. The other two input variables are assumed to be deterministic with values 0.8 and 2.4 respectively. The analysis results are summarized in Table 4. The computational costs (number of power loss calculations) are also listed in the Table.

The moment estimation results of all methods closely match with each other except the kurtosis calculated by 3^n FFNI and 3n UDR, which is due to the nonlinearity of the power loss function. The significance of this error obviously depends on the situation, our target of analysis, the confidence level required and so on. In the case that reliability is our target of analysis, the effect of error in moment estimation can be roughly calculated with the finite difference method using the Pearson system introduced in Section 2.1 under the assumption that the tail region PDF is well described by the Pearson system.

Table 4 Analysis results of engine piston example

	MCS	3^n FFNI	5^n FFNI	3n UDR
mean	0.3931	0.3920	0.3924	0.3920
Std	0.0311	0.0299	0.0300	0.0300
skewness	-0.5855	-0.5867	-0.5822	-0.5663
kurtosis	3.1041	3.0406	3.4336	3.1188
Pr[PL<0.3]	5.400E-03	3.684E-03	5.161E-03	4.117E-03
Pr[PL<0.45]	9.980E-01	9.996E-01	9.922E-01	9.972E-01
Fn call	100 k	9	25	5
	5n UDR	4^{th} PCE	FORM	
mean	0.3926	0.3924		
Std	0.0302	0.0300		
skewness	-0.5503	-0.58460		
kurtosis	3.4833	3.46129		
Pr[PL<0.3]	5.278E-03	5.700E-03	1.903E-03	
Pr[PL<0.45]	9.889E-01	1.000E-00	0.9981	
Fn call	9	25	(15, 30)	

In probability calculation results, it is seen that 5^n FFNI, 5n UDR, and the 4^{th} order PCE match well with the result by MCS while 3^n FFNI and 3n UDR and FORM underestimate the probability. It is natural that the errors in moment estimation result in errors in probability calculation. The error in the result of FORM is related to the nonlinearity of the power loss function.

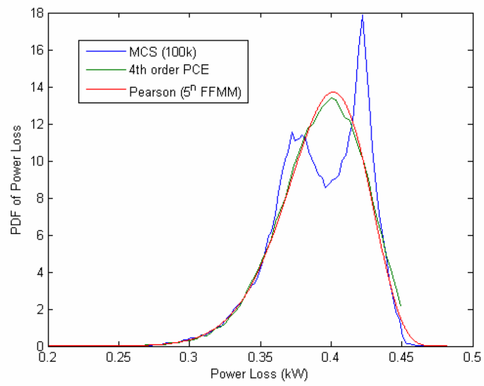


Figure 10. PDF of power loss function

In Fig. 10 the PDF of the power loss function is plotted. The PDF of the PCE is obtained by running MCS on the PCE, and the moments obtained by 5^n FFNI are used with the Pearson system to determine a PDF curve. We found that the actual PDF obtained by the MCS has an irregular shape with double peaks, but both the Pearson system based on the first four moments and the PCE fail to find the accurate shape of the PDF in this example. The Pearson system is not capable of representing such irregular shape and it is an inherent limitation of fitting a PDF using a small number of moments. This example further verifies the notion that the PDF estimated by the first few moments is not unique. The PCE can represent any square integrable random variable in theory, however, this problem shows that the convergence to the real PDF might be very slow in some cases.

3.4 Fortini's clutch

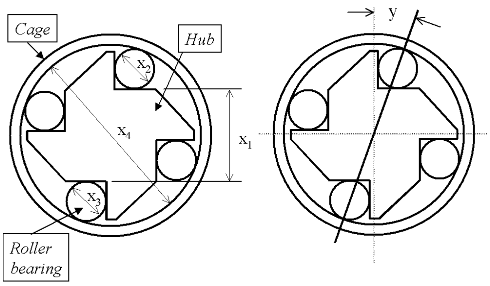


Figure 11. Fortini's clutch

The last example is the Fortini's clutch (Fig. 11) used in many tolerance analysis literature (Creveling 1997). The contact angle y is given in terms of the independent component variables, x_1, x_2, x_3 and x_4 as follows:

$$y = \arccos\left(\frac{x_1 + 0.5(x_2 + x_3)}{x_4 - 0.5(x_2 + x_3)}\right) \tag{21}$$

The moments and probabilities at the left tail region of y are calculated with all methods. The Pearson system is used with FFNI and also with UDR to calculate the probability, while the probability of PCE is calculated by MCS with 1,000 k samples. To see the effect of non-normality of input variables, two different input settings are tried, one with all normally distributed variables, the other with two non-normal variables.

3.4.1 Case 1: all normal inputs

The distribution parameters are summarized in Table 7, and the results of analysis are given in Table 8. All the methods calculate moments accurately except the UDR method. The skewness and kurtosis estimation of the 5n UDR is worse than that of 3^n FFNI. This means that there exist significant interaction effects between variables and it is verified by the analysis of variance (ANOVA) that the interaction between x_1 and x_4 is important. The same reason contributes to the errors in the probability estimation. The performance of FORM is very satisfactory in both accuracy and efficiency. The coefficients of PCE are calculated by 5^n FFNI and the results are as accurate as 5^n FFNI.

Table 7 Input random variables for Fortini's clutch example (case 1)

Component	Distribution type	Mean	Standard Deviation
x_1	Normal	55.29 mm	0.0793 mm
x_2	Normal	22.86 mm	0.0043 mm
x_3	Normal	22.86 mm	0.0043 mm
x_4	Normal	101.60 mm	0.0793 mm

Table 8 Uncertainty analysis results of Fortini's clutch example (case 1)

	MCS (1000k)	3^n FFNI	5^n FFNI	3n UDR
mean	0.1219	0.1219	0.1219	0.1219
std	0.0118	0.0118	0.0118	0.0117
skewness	-0.3193	-0.3082	-0.3157	-0.1436
kurtosis	3.2878	3.2000	3.2827	3.0000
Pr(y<4deg)	2.070E-04	1.257E-04	1.732E-04	1.789E-05
Pr(y<5deg)	4.690E-03	4.514E-03	4.830E-03	2.490E-03
Pr(y<6deg)	7.836E-02	7.904E-02	7.849E-02	7.482E-02
Fn call	1000k	81	625	9

Case 2: with 2 non-normal variables

The input x_1 is assumed to follow the beta distribution with parameters $\eta = \gamma = 5.0$ and x_4 is assumed to follow the Rayleigh distribution with the same means and standard deviations with case 1 (Table 9). The notations of distribution parameters follow those in (Hahn and Shapiro 1967). Results are summarized in Table 10.

Table 9 Input random variables for Fortini's clutch example

	Distribution	Mean	Standard Deviation	Parameters for non-normal distributions
x_1	Beta	55.29 mm	0.0793 mm	$\gamma_1 = \eta_1 = 5.0$
x_2	Normal	22.86 mm	0.0043 mm	$(55.0269 \le x_1 \le 55.5531)$
x_3	Normal	22.86 mm	0.0043 mm	$\hat{\sigma}_4 = 0.1211$
x_4	Rayleigh	101.60 mm	0.0793 mm	$(x_4 \ge 101.45)$

Table 10 Analysis results of Fortini's clutch example

	MCS	3^n FFNI	5^n FFNI	3n UDR
mean	0.1219	0.1219	0.1219	0.1219
std	0.0117	0.0117	0.0117	0.0116
skewness	-0.0516	-0.0497	-0.0530	0.0989
kurtosis	2.8810	2.8488	2.8827	2.8401
Pr(y<4deg)	0.000E+00	3.791E-07	1.058E-06	0.000E+00
Pr(y<5deg)	1.222E-03	1.241E-03	1.396E-03	3.707E-04
Pr(y<6deg)	7.381E-02	7.288E-02	7.272E-02	6.671E-02
Fn_call	1000k	81	625	9

	5n UDR	4^{th} PCE	FORM
mean	0.1219	0.1219	
std	0.0116	0.0117	
skewness	0.0964	-0.0577	
kurtosis	2.8662	2.8930	
Pr(y<4deg)	0.000E+00	0.000E+00	Diverge
Pr(y<5deg)	4.491E-04	1.220E-03	Diverge
Pr(y<6deg)	6.668E-02	7.402E-02	8.771E-02
Fn_call	17	625	31

The trends are almost similar with case 1, except that the HL-RF algorithm used in FORM has some difficulties in finding the MPP when the probability is small. The divergence occurs when the search point of HL-RF algorithm goes outside the domain where the non-normal variables are defined. FFNI shows consistently good results and similarly to the results of case 1 UDR shows some errors in high order moments and probability. However, we can see that the estimation of the mean and standard deviation is still good and there is no additional loss of accuracy caused by non-normality of inputs. The accuracy of PCE is still good although the inverse Rosenblatt transformation is used to transform x_1 and x_4 to the standard normal variables, which is different from the trend shown in examples 1 and 2.

4. DISCUSSIONS AND CONCLUSIONS

In the paper, several categories of uncertainty propagation techniques, including a few techniques that are receiving growing attentions, i.e., univariate dimension reduction and polynomial chaos expansion, are examined in depth to understand the characteristics and limitations of various methods. Comparative studies are performed using illustrative examples. Ideally, the performances should be evaluated under a considerable range of system nonlinearity, various distributions of input random variables, and various dimensionality in terms of accuracy and efficiency. Hence, it might not be plausible to judge or rank the methods with just several examples. However, through this comparison study, some characteristics, advantages and disadvantages of each method can be generalized.

It is noted that the FFNI and UDR methods use direct numerical integrations to obtain the statistical moments, and their accuracy depends on the integration order of the quadrature rule adopted. While the computational cost increases exponentially in FFNI, UDR has a linear increase of function evaluations with the number of input random variables. This outstanding efficiency is obtained by sacrificing accurate considerations of the interaction effects which may exist in the system. For examples, we could see that this approximation can cause errors especially in high order moments and the small probability at tail distribution. One nice feature of FFNI and UDR is that they are robust against the non-normality of inputs. Based on the four statistical moments obtained, the Pearson system of distribution can be further adopted to obtain the probability and the complete PDF. These methods work well when the PDF of a performance function has a regular shape with one mode, however, they are limited in finding the complete PDF with a few lower order moments.

The PCE, a functional expansion based method that gains popularity, is tested with coefficients calculated by the numerical integration and the inverse Rosenblatt transformation. The procedure of calculating the expansion coefficients is illustrated in detail which can be effectively applied to the black-box type functions. With the PCE approach, the accuracy and the computational cost depend on the integration scheme adopted to calculate the coefficients. The implementation of the PCE used in this study is subject to two major sources of error. The first one is the truncation error with the finite expansion, the second is the coefficient estimation error related with the numerical integration scheme to calculate the expectation. A further study about the effect of the coefficient estimation errors on the convergence behavior of PCE seems necessary. The transformation used to treat non-normal inputs can degrade the accuracy which makes this approach less robust against the non-normality of inputs compared to the FFNI and UDR methods. On the other hand, this feature of obtaining directly the PDF function of an output response using the PCE method is very useful.

Related to the different scenarios of design under uncertainty, for example, robust design, reliability-based design, and utility optimization, a method for UP should be selected based on the required level of uncertainty quantification (e.g., low-order moments, tail probability, and complete PDF), accuracy or confidence level, as well as the computational cost or efficiency. It is observed that if there is no significant interaction between variables, the UDR method is the most efficient method for moment estimations. The UDR method is flexible with the type of input distributions. Its performance in assessing tail probability is comparable with that of the MPP based method, sometimes even better, but not always. The MPP based method is an efficient approach to the evaluation of tail probability. However, it appears to be the least stable method that is sensitive to the type of input distributions and the function nonlinearity. The PCE method is a useful approach when a complete PDF description is desired. Its performance in evaluating the statistical moments is comparable to the FFNI approach. With the current transformation method, the performance of the PCE method deteriorates when non-normal input distributions are considered. On the other hand, the FFNI method is flexible with the type of input distributions.

One aspect which is yet to be examined in this work is the performance of the PCE with respect to different sample sizes. The other aspect which was not investigated in this study is the effect of correlations among input variables on the performance of each method. All the methods except MCS need transformation of those correlated variables into uncorrelated variables. Performances of hybrid approaches that combine multiple techniques including the metamodeling technique are yet studied to make the current scope of comparison manageable.

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APPENDIX

Table A1 PCE with coefficients calculated analytically

k	Polynomial chaos expansion
1	$y^{(p)} = 1 + 0.2\xi$
2	$y^{(p)} = 1.04 + 0.4\xi + 0.04\left(\xi^2 - 1\right)$
3	$y^{(p)} = 1.12 + 0.6240\xi + 0.12\left(\xi^2 - 1\right) + 0.008\left(\xi^3 - 3\xi\right)$
4	$y^{(p)} = 1.2448 + 0.8960\xi + 0.2496\left(\xi^2 - 1\right) + 0.0320\left(\xi^3 - 3\xi\right) + 0.0016\left(\xi^4 - 6\xi^2 + 3\right)$
5	$y^{(p)} = 1.4240 + 1.2448\xi + 0.4480\left(\xi^2 - 1\right) + 0.0832\left(\xi^3 - 3\xi\right) + 0.0080\left(\xi^4 - 6\xi^2 + 3\right)$
6	$y^{(p)} = 1.6730 + 1.7088\xi + 0.7469\left(\xi^2 - 1\right) + 0.1792\left(\xi^3 - 3z\right) + 0.0250\left(\xi^4 - 6\xi^2 + 3\right)$
7	$y^{(p)} = 2.0147 + 2.3421\xi + 1.3348\left(\xi^2 - 1\right) + 0.3485\left(\xi^3 - 3\xi\right) + 0.0672\left(\xi^4 - 6\xi^2 + 3\right)$

Table A2 PCE with coefficients calculated numerically

k	Polynomial chaos expansion
1	$y^{(p)} = 1 + 0.2\xi$
2	$y^{(p)} = 1.04 + 0.4\xi + 0.04\left(\xi^2 - 1\right)$
3	$y^{(p)} = 1.12 + 0.6240\xi + 0.12\left(\xi^2 - 1\right) + 0.008\left(\xi^3 - 3\xi\right)$
4	$y^{(p)} = 1.2448 + 0.8960\xi + 0.2496\left(\xi^2 - 1\right) + 0.0320\left(\xi^3 - 3\xi\right) + 0.0016\left(\xi^4 - 6\xi^2 + 3\right)$
5	$y^{(p)} = 1.4240 + 1.2448\xi + 0.4480\left(\xi^2 - 1\right) + 0.0832\left(\xi^3 - 3\xi\right) + 0.0080\left(\xi^4 - 6\xi^2 + 3\right)$
6	$y^{(p)} = 1.6730 + 1.7088\xi + 0.7469\left(\xi^2 - 1\right) + 0.1792\left(\xi^3 - 3\xi\right) + 0.0246\left(\xi^4 - 6\xi^2 + 3\right)$
7	$y^{(p)} = 2.0147 + 2.3421\xi + 1.1962\left(\xi^2 - 1\right) + 0.3483\left(\xi^3 - 3\xi\right) + 0.0604\left(\xi^4 - 6\xi^2 + 3\right)$