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Post-Inflationary Gravitino Production Revisited

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ABSTRACT

We revisit gravitino production following inflation. As a first step, we review the standard calculation of gravitino production in the thermal plasma formed at the end of post-inflationary reheating when the inflaton has completely decayed. Next we consider gravitino production prior to the completion of reheating, assuming that the inflaton decay products thermalize instantaneously while they are still dilute. We then argue that instantaneous thermalization is in general a good approximation, and also show that the contribution of non-thermal gravitino production via the collisions of inflaton decay products prior to thermalization is relatively small. Our final estimate of the gravitino-to-entropy ratio is approximated well by a standard calculation of gravitino production in the post-inflationary thermal plasma assuming total instantaneous decay and thermalization at a time $t \simeq 1.2/\Gamma_\phi$. Finally, in light of our calculations, we consider potential implications of upper limits on the gravitino abundance for models of inflation, with particular attention to scenarios for inflaton decays in supersymmetric Starobinsky-like models.

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1 Introduction

Measurements by the Planck satellite [1] and ground-based experiments such as BICEP2/Keck Array [2, 3] are probing the cosmic microwave background (CMB) ever more precisely, a new round of experiments is on the way, and far more accurate experiments are being proposed for the future. The data are already putting strong pressure on models of cosmological inflation, excluding many and constraining severely the survivors. The most stringent constraints are generally those from the magnitude and tilt, n_s, of the scalar perturbation spectrum, and the tensor-to-scalar ratio, r, with constraints on non-Gaussianities, isocurvature perturbations, etc., being less powerful in the context of most slow-roll models [4]. The predictions of these models depend, in general, on the number of e-folds of inflation, N_*, which depends in turn of the amount of reheating at the end of inflation [5, 6].

Comparisons of the Planck data with inflationary models typically consider $40 < N_* < 60$, and characteristic model predictions for n_s, in particular, vary by amounts comparable with the 68% experimental range in n_s as N_* varies over this range. This means that the experimental data are already starting to provide interesting constraints on N_*, and hence indirectly on the amount of reheating. The latter depends, in turn, on the decay rate of the inflaton into relativistic particles. For example, if the dominant inflaton decay is into two particles, reheating and hence the required number of e-folds N_* and the predictions for n_s and r depend on the two-body decay coupling y.

Conversely, the experimental constraints on n_s and r can be used to constrain the coupling y in the context of any specific model. For example, in models whose predictions resemble those of the Starobinsky R^2 model [7], a combination of the Planck, BICEP2/Keck Array and BAO data yields $N_* \gtrsim 50$ at the 68% CL, corresponding to $y \gtrsim 5 \times 10^{-8}$ [6].

In supersymmetric models, cosmological and astrophysical constraints on the abundance of gravitinos produced after inflation yield complementary restrictions on the amount of reheating, and hence y [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. These constraints may arise from considerations of the relic dark matter density due to gravitinos or their decay products and/or from limits on late-decaying gravitinos imposed, e..g., by the success of Big-Bang nucleosynthesis calculations [26, 27, 16, 22]. These upper limits on the produced gravitino abundance translate into an upper bound on a two-body coupling of $y \lesssim 10^{-5}$ [6].

In view of the present and prospective future constraints on inflaton decay via CMB limits on N_* and the competition (in supersymmetric models) with constraints from the gravitino abundance, in this paper we revisit the issue of gravitino production following inflation. One source of gravitinos that is well understood is production by particle collisions in the thermal plasma that fills the Universe after reheating [13, 15, 16, 18, 19, 20, 23, 24, 25]. However, gravitinos could also have been produced by particle collisions before the reheating process was complete, either by collisions of relativistic inflaton decay products before thermalization, or in any dilute thermal plasma formed by their collisions while inflaton decay was continuing [19, 25].

We consider all these mechanisms in this paper, [1] and compare the naive approximation of total instantaneous decay at $t = c/\Gamma_\phi$, where Γ_ϕ is the inflaton decay rate and c is some $\mathcal{O}(1)$ constant, with the exact solution involving continuous decays producing the thermal bath. Whilst the common approximation neglects gravitino production at times $t < c/\Gamma_\phi$, it also neglects the dilution of the gravitino abundance due to radiation produced at $t > c/\Gamma_\phi$. Remarkably, the approximate and exact solutions agree for the choice $c = 1.2$ [2]. Using the production rate for gravitinos, we then derive an analytic expression for later cosmological evolution assuming continuous inflaton decays.

We apply these results to recent inflationary models based on no-scale supergravity [35, 36, 37, 38, 39]. In particular, we focus on the phenomenological aspects of Starobinsky-like models of inflation [39]. Reheating in no-scale models does not occur automatically [40], but instead requires either an explicit coupling of the inflaton to matter [35, 37, 39], a coupling to moduli [39], or a coupling to the gauge sector through the gauge kinetic function [40, 39]. Direct decays of the inflaton to gravitinos may, in general, compete with the thermal production, and both should be considered when setting limits on the couplings governing inflaton decay.

The standard calculation of gravitino production in the thermal plasma formed after reheating is reviewed in Section 2. Then, in Section 3 we calculate gravitino production before the completion of reheating, assuming that the inflaton decay products thermalize instantaneously. These are, in some sense, opposite extremes for the treatment of gravitino production. We also provide an analytic solution to the late time gravitino yield when the instantaneous decay approximation is dropped. In Section 4 we analyze the degree to which instantaneous thermalization is a good approximation, and also discuss non-thermal contributions to the gravitino production rate prior to thermalization. In Section 5 we review the consequences for inflaton couplings and compare with the constraints from Big Bang Nucleosynthesis, the relic cold dark matter density, and the CMB constraints on the number of inflationary e-folds. In particular, we derive constraints on the couplings governing inflaton decay in phenomenological no-scale inflation models [39, 6]. Finally, in Section 6 we summarize our conclusions.

2   Production in the Thermal Plasma after Reheating

Most calculations of gravitino production assume the instantaneous decay of the inflaton at $t = 1/\Gamma_\phi$ where Γ_ϕ is the inflaton decay rate. Since the Universe is dominated by the oscil-

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[1] On the other hand, we disregard the possible additional gravitino quanta that could be produced if the inflaton experiences a strong non-perturbative decay at the onset of its oscillations, also known as preheating, since non-perturbative gravitino production at preheating has been shown to be small [28, 29, 19, 30, 31, 32, 33]. In this scenario, gravitinos may also be perturbatively produced by the non-thermal distributions formed at preheating. We disregard this effect since it is model-dependent, and since the evolution of these distributions is not well understood [34].

[2] We note that two choices of c are commonly found in the literature, namely $c = 1$, which is relatively accurate, and $c = 2/3$, corresponding to the condition $\Gamma_\phi = H$, where H is the Hubble parameter. As shown in [24, 25], the latter choice yields a gravitino abundance that is too large by a factor $\sim \sqrt{3/2}$.

lations of the inflaton prior to decay, and these oscillations act as matter, this is equivalent to $\Gamma_\phi = 3H/2$. It is then assumed that the decay products thermalize very rapidly [41]. In this case, one can define a reheating temperature from the instantaneous conversion of the energy density in oscillations to that of radiation,

$$T_{\text{reh}} = \left(\frac{30\rho_{\text{reh}}}{\pi^2 g_{\text{reh}}} \right)^{1/4}, \tag{1}$$

where ρ_{reh} denotes the energy density of radiation and g_{reh} denotes the effective number of degrees of freedom at the 'moment' of reheating. Gravitinos are then produced by scattering processes in the thermalized radiation-filled Universe, after which they decay with the following decay rate into particles within the minimal supersymmetric extension of the Standard Model (MSSM) [17],

$$\Gamma_{3/2} = \frac{193}{384\pi} \frac{m_{3/2}^3}{M_P^2}. \tag{2}$$

Here M_P refers to the reduced Planck mass, $M_P = (8\pi G_N)^{-1/2} \simeq 2.4 \times 10^{18}\,\text{GeV}$. Since the interactions of gravitinos are very weak, they do not thermalize with the radiation background. Therefore, the Boltzmann equation for the gravitino number density $n_{3/2}$ can be written as

$$\frac{dn_{3/2}}{dt} + 3Hn_{3/2} = \langle \sigma_{\text{tot}} v_{\text{rel}} \rangle n_{\text{rad}}^2 - \frac{m_{3/2}}{\langle E_{3/2} \rangle} \Gamma_{3/2} n_{3/2}, \tag{3}$$

where $\langle \sigma_{\text{tot}} v_{\text{rel}} \rangle$ is the thermally-averaged gravitino production cross section, $n_{\text{rad}} = \zeta(3)T^3/\pi^2$ is the number density of any single bosonic relativistic degree of freedom, and $\langle E_{3/2} \rangle / m_{3/2}$ is the averaged Lorentz factor. Inverse-scattering terms are omitted, because their contributions are unimportant at the reheating temperatures of relevance [16].

The thermally-averaged cross section for the Standard Model $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group was calculated in [20, 23, 25]. Including contributions from $2 \to 2$ gauge scatterings, production via $1 \to 2$ decays allowed by thermal masses, and the effect of the top Yukawa coupling y_t, it can be parametrized as

$$\langle \sigma_{\text{tot}} v_{\text{rel}} \rangle = \langle \sigma_{\text{tot}} v_{\text{rel}} \rangle_{\text{gauge}} + \langle \sigma_{\text{tot}} v_{\text{rel}} \rangle_{\text{top}} \tag{4}$$

with

$$\langle \sigma_{\text{tot}} v_{\text{rel}} \rangle_{\text{top}} = 1.29 \frac{|y_t|^2}{M_P^2} \left[1 + \frac{A_t^2}{3m_{3/2}^2} \right], \tag{5}$$

where A_t is the top-quark supersymmetry-breaking trilinear coupling, and

$$
\begin{aligned}
\langle \sigma_{\text{tot}} v_{\text{rel}} \rangle_{\text{gauge}} &= \sum_{i=1}^{3} \frac{3\pi c_i g_i^2}{16\zeta(3)M_P^2} \left[1 + \frac{m_{\tilde{g}_i}^2}{3m_{3/2}^2} \right] \ln \left(\frac{k_i}{g_i} \right) \\
&= \frac{26.24}{M_P^2} \left[\left(1 + 0.558 \frac{m_{1/2}^2}{m_{3/2}^2} \right) - 0.011 \left(1 + 3.062 \frac{m_{1/2}^2}{m_{3/2}^2} \right) \log \left(\frac{T_{\text{reh}}}{10^{10}\,\text{GeV}} \right) \right], \tag{6}
\end{aligned}
$$

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Gauge group	g_i	c_i	k_i
$U(1)_Y$	g'	9.90	1.469
$SU(2)_L$	g	20.77	2.071
$SU(3)_c$	g_s	43.34	3.041

Table 1: *The values of the constants c_i and k_i in the parameterization (6) for the Standard Model gauge groups $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$. The values are obtained from a phenomenological fit to the result of [25], as explained in the text.*

where the $m_{\tilde{g}_i}$ are the gaugino masses and the constants c_i, k_i depend on the gauge group, as shown in Table 1. We have obtained these values through a phenomenological fit to the result of [25] using the convenient parametrization of [23], under the assumption of a unified gauge coupling $\alpha = 1/24$ and universal gaugino masses $m_{1/2}$ at the scale $M_{\mathrm{GUT}} = 2 \times 10^{16}\,\mathrm{GeV}$. Table 1 differs from the result of [23] in that it includes gravitino production via decays, leading to a production rate that is about twice larger. We have included in (6) the leading logarithmic corrections to the running of all the gauge couplings and gaugino masses, which at one-loop order are given by

$$g_i(T)^2 = \frac{g_i(M_{\mathrm{GUT}})^2}{1 - \frac{b_i}{8\pi^2}\, g_i(M_{\mathrm{GUT}})^2\, \ln(T/M_{\mathrm{GUT}})}\,, \qquad \begin{pmatrix} b' \\ b \\ b_s \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \\ -3 \end{pmatrix}\,, \qquad (7)$$

$$m_{\tilde{g}_i}(T) = \left(\frac{g_i(T)}{g_i(M_{\mathrm{GUT}})}\right)^2 m_{1/2}\,. \qquad (8)$$

It is worth noting that the first term in the gaugino mass-dependent factors $(1 + m_{\tilde{g}_i}^2/3m_{3/2}^2)$ corresponds to the production of the transversally polarized gravitino, while the second term is associated with the production of the longitudinal (Goldstino) component.

Immediately after reheating, the second term in the r.h.s. of (3) is negligible. In terms of the gravitino yield

$$Y_{3/2} \equiv \frac{n_{3/2}}{n_{\mathrm{rad}}}\,, \qquad (9)$$

the Boltzmann equation (3) can be rewritten in the form

$$\dot{Y}_{3/2} + 3\left(H + \frac{\dot{T}}{T}\right) Y_{3/2} = \langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}\,. \qquad (10)$$

Under the assumption of entropy conservation, $gT^3 a^3 = \mathrm{const.}$, where a is the cosmological scale factor. Eq. (10) is equivalent to

$$\frac{dY_{3/2}}{dT} - \frac{d\ln g}{dT} Y_{3/2} = -\frac{\langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}}{HT}\left[1 + \frac{T}{3}\frac{d\ln g}{dT}\right]\,. \qquad (11)$$

Straightforward integration then yields

$$Y_{3/2}(T) = Y_{3/2}(T_{\mathrm{reh}})\frac{g(T)}{g(T_{\mathrm{reh}})} - g(T)\int_{T_{\mathrm{reh}}}^{T} \frac{\langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}(\tau)}{g(\tau)H(\tau)\tau}\left[1 + \frac{\tau}{3}\frac{d\ln g(\tau)}{d\tau}\right] d\tau\,. \qquad (12)$$

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Notice that this integration begins at T_{reh}, consistent with the assumption that inflaton decay and thermalization are instantaneous and simultaneous at T_{reh}, and runs to lower T. Assuming a vanishing abundance at T_{reh}, and disregarding the weak dependence on temperature of the integrand in the r.h.s., integration from T_{reh} to $T \ll T_{\mathrm{reh}}$ yields

$$Y_{3/2}(T) \simeq \frac{\langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}(T_{\mathrm{reh}})}{H(T_{\mathrm{reh}})} \times \frac{g(T)}{g(T_{\mathrm{reh}})} \,. \tag{13}$$

Hence the final abundance of gravitinos is given by the ratio of the production rate ($\langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}$) to the Hubble rate at reheating, diluted by subsequent particle annihilations and accounting for the ratio of numbers of degrees of freedom at T to that at T_{reh}.

During the radiation-dominated era, the cosmic time and temperature are related by

$$t = \sqrt{\frac{45}{2\pi^2 g}} \frac{M_P}{T^2} \,. \tag{14}$$

When the temperature of the Universe drops to $T \ll (\Gamma_{3/2}/\Gamma_\phi)^{1/3} T_{\mathrm{reh}}$, where Γ_ϕ is the inflaton decay rate, the decay term in the Boltzmann equation (3) dominates over the scattering term. In this case, gravitinos have redshifted their momenta away, which implies that (3) may be rewritten as

$$\dot{Y}_{3/2} = -\Gamma_{3/2} Y_{3/2} \,. \tag{15}$$

Under the assumption that $m_{3/2} \ll 10^{13}\,\mathrm{GeV} \times (T_{\mathrm{reh}}/10^{10}\mathrm{GeV})$, the approximation (13) may be taken as an initial condition for (15). Denoting $g_{\mathrm{reh}} = g(T_{\mathrm{reh}})$, the gauge contribution to the gravitino abundance can finally be written as

$$Y_{3/2}(T) \simeq \frac{\langle \sigma_{\mathrm{tot}} v_{\mathrm{rel}} \rangle n_{\mathrm{rad}}(T_{\mathrm{reh}})}{H(T_{\mathrm{reh}})} \times \frac{g(T)}{g_{\mathrm{reh}}} \times e^{-\Gamma_{3/2}t} \,,$$

$$\simeq 7.40 \times 10^{-10} e^{-\Gamma_{3/2}t} \frac{g(T)}{g_{\mathrm{reh}}^{3/2}} \left(\frac{T_{\mathrm{reh}}}{10^{10}\,\mathrm{GeV}} \right) \sum_{i=1}^{3} c_i \, g_i(T_{\mathrm{reh}})^2 \left(1 + \frac{m_{\tilde{g}_i}(T_{\mathrm{reh}})^2}{3m_{3/2}^2} \right) \ln \left(\frac{k_i}{g_i(T_{\mathrm{reh}})} \right)$$

$$\simeq 3.96 \times 10^{-8} e^{-\Gamma_{3/2}t} \frac{g(T)}{g_{\mathrm{reh}}^{3/2}} \left(\frac{T_{\mathrm{reh}}}{10^{10}\,\mathrm{GeV}} \right)$$

$$\times \left[\left(1 + 0.558 \frac{m_{1/2}^2}{m_{3/2}^2} \right) - 0.011 \left(1 + 3.062 \frac{m_{1/2}^2}{m_{3/2}^2} \right) \ln \left(\frac{T_{\mathrm{reh}}}{10^{10}\,\mathrm{GeV}} \right) \right] \,, \tag{16}$$

or

$$Y_{3/2}(T) \simeq 4.48 \times 10^{-11} e^{-\Gamma_{3/2}t} \left(\frac{T_{\mathrm{reh}}}{10^{10}\,\mathrm{GeV}} \right)$$

$$\times \left[\left(1 + 0.558 \frac{m_{1/2}^2}{m_{3/2}^2} \right) - 0.011 \left(1 + 3.062 \frac{m_{1/2}^2}{m_{3/2}^2} \right) \ln \left(\frac{T_{\mathrm{reh}}}{10^{10}\,\mathrm{GeV}} \right) \right] \,. \tag{17}$$

when we use $g_{\mathrm{reh}} = 915/4$ and $g(T \ll 1\,\mathrm{MeV}) = 3.91$. Fig. 1 compares the approximate result (17) and the yield obtained from integrating (10) numerically with the full one-loop

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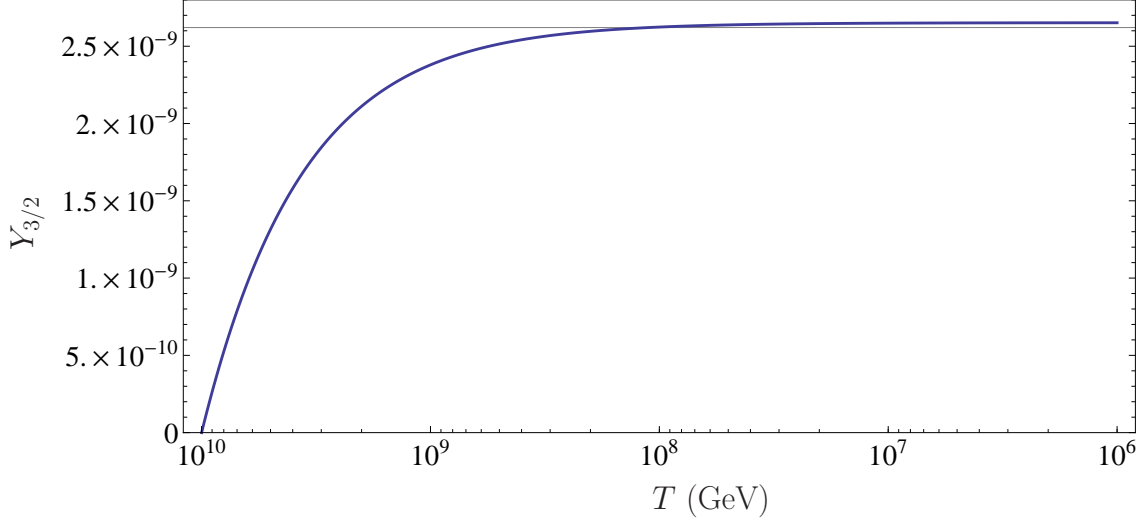Figure 1: *The gravitino yield $Y_{3/2}$ for $T_{\rm reh} = 10^{10}$ GeV as a function of temperature during radiation domination following the completion of reheating. The blue curve corresponds to the numerical integration of (10) under the simplifying assumption $g = 915/4 =$const., disregarding the top Yukawa contribution and the $m_{1/2}$-dependent Goldstino component. The approximation (16), not including the dilution factor $g(T)/g_{\rm reh}$, is displayed as the horizontal black line.*

correction to the coupling constants g_i. We see that the agreement is excellent, within $\sim 10\%$ for $T \gtrsim 10^9$ GeV, after thermalization is completed.

The yield in (17) can be rewritten in terms of Γ_ϕ once we specify the 'moment' of inflaton decay. For example, If we assume that the decay occurs instantaneously at $\Gamma_\phi t = 2\Gamma_\phi/3H = c$, where c is a constant that is $\mathcal{O}(1)$, then we can write

$$T_{\rm reh} = \left(\frac{40}{g_{\rm reh}\pi^2}\right)^{1/4} \left(\frac{\Gamma_\phi M_P}{c}\right)^{1/2} . \tag{18}$$

We see that the dependence of (18) on the arbitrary parameter c introduces an uncertainty in the gravitino abundance, when expressed in terms of the physical decay rate of the inflaton. Inserting (18) into (17), and disregarding the logarithmic correction, one has

$$Y_{3/2} \simeq \frac{0.00398}{\sqrt{c}} \left(\frac{\Gamma_\phi}{M_p}\right)^{1/2} \left(1 + 0.558 \frac{m_{1/2}^2}{m_{3/2}^2}\right) {\rm e}^{-\Gamma_{3/2} t} . \tag{19}$$

As the inflaton does not decay instantaneously, there is in fact no 'correct' value for c. Furthermore, the result (19) also assumes that all of the entropy produced by inflaton decays is already present at $t = c/\Gamma_\phi$. As we will see below, in fact only about 1/3 of the entropy produced by inflaton decays would have been released when $\Gamma_\phi t = 1$. Thus, an accurate determination of the thermal gravitino yield after inflation requires the integration of the coupled inflaton/radiation equations of motion.

To summarize this Section: using the thermal production rate computed in [25], we have provided in Fig. 1 a numerical solution to gravitino yield as a function of time, as well as an analytical solution to the late-time yield. In the following Sections we study the extent to which gravitino production before the completion of thermalization modifies this standard calculation.

3 Gravitino Production Assuming Instantaneous Thermalization of the Inflaton Decay Products

As discussed above, the previous calculation for the gravitino yield implicitly assumes the instantaneous decay of the inflaton at $\Gamma_\phi t \simeq 1$ and the instantaneous thermalization of the inflaton decay products. In reality, the decay is a continuous process and we now calculate the effect of gravitino production before reheating is complete, i.e., while the inflaton is still decaying, and still has a non-zero density ρ_ϕ. We assume initially that the decay products of the inflaton ϕ thermalize instantaneously, and discuss later the validity of this assumption. With this assumption, the Universe contains, in addition to the undecayed inflaton fraction, a dilute thermal plasma whose temperature is

$$T = \left(\frac{30\rho_\gamma}{\pi^2 g(T)} \right)^{1/4} , \tag{20}$$

where ρ_γ denotes the instantaneous energy density of the relativistic decay products. The time evolution of the full energy density during reheating is determined by the equations

$$\ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} + V_\phi = 0 , \tag{21}$$

$$\dot{\rho}_\gamma + 4H\rho_\gamma = \Gamma_\phi \rho_\phi , \tag{22}$$

$$\rho_\phi + \rho_\gamma = 3M_P^2 H^2 , \tag{23}$$

Reheating is complete when the energy density of the inflaton is negligible with respect to the density of the decay products,

$$\Omega_\gamma \equiv \frac{\rho_\gamma}{\rho_\phi + \rho_\gamma} = 1 - \delta . \tag{24}$$

for some suitable $\delta \ll 1$. Since the total gravitino abundance is always relatively small, we can neglect the contribution of its decays to ρ_γ. Moreover, since the oscillations of the inflaton about its minimum are much more rapid than any other time-scale in the problem, in particular since $m \gg \Gamma_\phi$, Eq. (21) can be approximated by averaging the energy density of the inflaton over the oscillations. For the matter-like oscillations of the inflaton, $\langle \rho_\phi \rangle = \langle \dot{\phi}^2/2 \rangle + \langle V \rangle \simeq \langle \dot{\phi}^2 \rangle$, and therefore (21) reduces approximately to

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi . \tag{25}$$

The solutions for $\rho_{\phi,\gamma}$ in terms of the scale factor are then given by

$$\rho_\phi(t) = \rho_{\text{end}} \left(\frac{a(t)}{a_{\text{end}}} \right)^{-3} e^{-\Gamma_\phi(t-t_{\text{end}})} , \tag{26}$$

$$\rho_\gamma(t) = \rho_{\text{end}} \left(\frac{a(t)}{a_{\text{end}}} \right)^{-4} \int_{\Gamma_\phi t_{\text{end}}}^{\Gamma_\phi t} \left(\frac{a(t')}{a_{\text{end}}} \right) e^{u_{\text{end}}-u} \, du . \tag{27}$$

where the subscript indicates that the quantity is evaluated at the end of inflation, which is defined by the condition for the equation-of-state parameter $w \equiv p/\rho = -1/3$ or equivalently when $\dot{\phi}_{\text{end}}^2/2 = V(\phi_{\text{end}})$. We note that ρ_{end} is model-dependent and that $u_{\text{end}} = \Gamma_\phi t_{\text{end}}$. In the case of a Starobinsky-like potential [7]

$$V(\phi) = \frac{3}{4}m^2 \left(1 - e^{-\sqrt{\frac{2}{3}}\phi} \right)^2 \tag{28}$$

we find $\rho_{\text{end}}/m^2 M_P^2 = 0.175$.

The exact solution of the Friedmann equation during reheating is given by

$$\rho(t) = \rho_{\text{end}} \left(1 + \sqrt{\frac{3}{4}\rho_{\text{end}}} \, (1+\bar{w}) \left(\frac{t-t_{\text{end}}}{M_P} \right) \right)^{-2} \tag{29}$$

or, equivalently, by

$$H(t) = \left(\frac{3}{2}(1+\bar{w})(t-t_{\text{end}}) + H_{\text{end}}^{-1} \right)^{-1} , \tag{30}$$

where the time-averaged equation-of-state parameter $\bar{w}$ of the radiation/inflaton fluid is defined as

$$\bar{w}(t) \equiv \frac{1}{t-t_{\text{end}}} \int_{t_{\text{end}}}^{t} w(t') \, dt' . \tag{31}$$

For a slowly-varying $\bar{w}(t)$, the scale factor at times $t > t_{\text{end}}$ can be approximated as

$$\frac{a(t)}{a_{\text{end}}} = \exp \left[\int_{t_{\text{end}}}^{t} H(t') \, dt' \right] \simeq \left(1 + \sqrt{\frac{3}{4}\rho_{\text{end}}}(1+\bar{w}) \left(\frac{t-t_{\text{end}}}{M_P} \right) \right)^{\frac{2}{3(1+\bar{w})}} . \tag{32}$$

During the initial stages of inflaton decay and thermalization, scalar field oscillations of the inflaton dominate and the equation-of-state parameter $w \approx 0$. Substituting (32) into (27), and introducing the quantities

$$v \equiv \Gamma_\phi(t-t_{\text{end}}) , \qquad A \equiv \frac{\Gamma_\phi}{m} \left(\frac{3}{4} \frac{\rho_{\text{end}}}{m^2 M_P^2} \right)^{-1/2} , \tag{33}$$

where m denotes the inflaton mass, we can write the energy density of the relativistic decay products at early times as

$$\rho_\gamma \simeq \rho_{\text{end}} \left(\frac{v}{A} + 1 \right)^{-8/3} \int_0^v \left(\frac{v'}{A} + 1 \right)^{2/3} e^{-v'} \, dv' \quad , \quad v \ll 1 . \tag{34}$$

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The solution (34) predicts a maximum of the energy density of the decay products: for $A \ll 1$ it corresponds to

$$v_{\text{max}} \simeq 0.80A \qquad \Rightarrow \qquad \rho_{\gamma,\text{max}} \simeq 0.21A\rho_{\text{end}} \,. \qquad (35)$$

This in turn implies a maximum temperature of the dilute plasma after the start of inflaton decay:

$$T_{\text{max}} = 0.89 \left(\frac{A\rho_{\text{end}}}{g(T_{\text{max}})} \right)^{1/4} \simeq 0.74 \left(\frac{\Gamma_\phi m M_P^2}{g_{\text{max}}} \right)^{1/4} \,, \qquad (36)$$

where the second equality uses the value of ρ_{end} for the Starobinsky potential. Comparing T_{max} with the reheat temperature defined in (18), we have

$$\frac{T_{\text{max}}}{T_{\text{reh}}} = 0.52 \left(\frac{g_{\text{reh}} m}{g_{\text{max}} \Gamma_\phi} \right)^{1/4} \,, \qquad (37)$$

which correctly accounts for the peak of the temperature evolution shown in Fig. 2, when the inflaton mass is taken to be $m \simeq 10^{-5} M_P$.
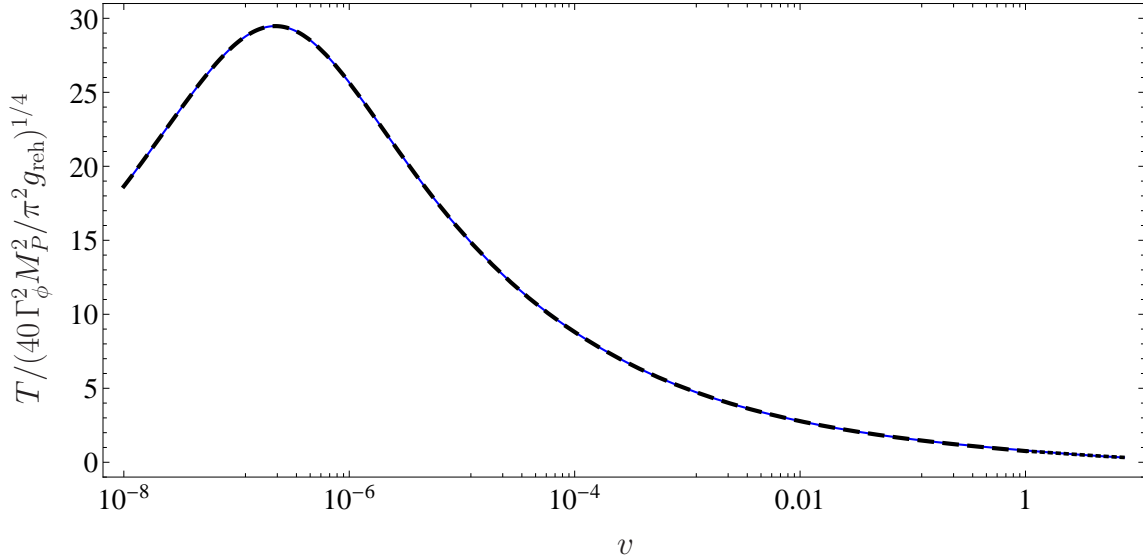

Figure 2: *The temperature of the dilute plasma that would be formed if the inflaton decay products thermalized instantaneously, as a function of $v = \Gamma_\phi(t - t_{\text{end}})$, for $\Gamma_\phi/M_P = 10^{-12}$ and $A \simeq 2.76 \times 10^{-7}$ (corresponding to the Starobinsky-like potential (28)). The numerical solution of equations (21)-(23) is shown as the solid blue curve. The dashed (dotted) curve corresponds to the approximate solution (38) for $v < 1$ ($v > 1$). All results are normalized relative to the reheating temperature derived from (39). We note the presence of the peak in the temperature at $v_{\text{max}} \simeq 2.2 \times 10^{-7}$, see Eq. (35).*

During later stages of decay and reheating, the equation-of-state parameter of the radiation/inflaton fluid can be approximated by the average $\bar{w}(t_{\text{reh}})$, where t_{reh} is the time at

which reheating completes. To be more precise, we denote by $t_{\rm reh}$ the time at which the fractional energy density remaining in the inflaton field falls to a value $\delta \ll 1$, see Eq. (24). The functional dependence of $t_{\rm reh}$ on δ, as well as the value of $\bar{w}$, can be obtained through the iterative procedure described in [6]. Under the reasonable assumption that $A \ll 1$, one obtains $\Gamma_\phi (t_{\rm reh} - t_{\rm end}) \equiv v_{\rm reh} \simeq 0.655 - 1.082 \ln \delta$, and $\bar{w} \simeq 0.273$ for $\delta = 0.002$ [6]. One can then substitute (32) with $w \simeq \bar{w}$ into (27) to obtain ρ_γ at late times. We can take into account the initial condition by matching the solution for $w = 0$ (34) with the late solution for $w = \bar{w}$. Choosing the matching point at $v = 1$ (corresponding to $t - t_{\rm end} = \Gamma_\phi^{-1}$), we obtain

$$
\frac{\rho_\gamma}{\rho_{\rm end}} \simeq \begin{cases} A^2 e^A (v + A)^{-8/3} \left[\boldsymbol{\gamma}(\tfrac{5}{3}, v + A) - \boldsymbol{\gamma}(\tfrac{5}{3}, A) \right] , & v < 1 , \\[2mm] 1.44 (1 + \bar{w})^{-2} A^2 v^{-8/3(1+\bar{w})} \boldsymbol{\gamma}(\tfrac{5+3\bar{w}}{3(1+\bar{w})}, v) , & v > 1 , \end{cases} \tag{38}
$$

where $\boldsymbol{\gamma}$ denotes the lower incomplete gamma function. The energy density at the end of reheating (more accurately, at $t = t_{\rm reh}$ defined above) may be evaluated from (29), resulting in

$$
\rho_{\rm reh} \simeq \frac{4}{3} \left(\frac{M_P \Gamma_\phi}{(1 + \bar{w}) v_{\rm reh}} \right)^2 . \tag{39}
$$

Fig. 2 shows the temperature evolution during reheating obtained from the approximate solution (38) and from the numerical solution of equations (21)-(23), where the scalar potential has been chosen as the Starobinsky potential: we see excellent agreement.

For a brief period of time, the temperature is significantly larger than $T_{\rm reh}$ and, as seen in (37), the relative increase in T scales with m/Γ_ϕ. Since the production rate of gravitinos is proportional to T, one might suspect that gravitino production at $v < 1$ could contribute substantially to the total gravitino abundance. However, one must also take into account the growth of the entropy $S = sa^3$ during the epoch of inflaton decay and reheating illustrated in Fig. 3, where we show the entropy normalized to its final value $S_{\rm final}$. We note that the entropy is relatively small during the period of the bump in temperature seen in Fig. 2. Although the plasma is hot at this time, any production at $v < 1$ will be diluted subsequently by later inflaton decays that builds up entropy, as seen in Fig. 3. Thus, the net effect of the increased temperature and later dilution is not obvious a priori.

Gravitinos are produced continuously by collisions of the relativistic decay products throughout this reheating epoch. However, the gravitino yield cannot be computed directly from (11), due to the continuing injection of entropy into the plasma while the decay of the inflaton continues, as illustrated in Fig 3. We therefore make use of the full equation (10), which can be rewritten in terms of v as

$$
Y_{3/2}' + 3 \left(\hat{H} + \frac{\hat{T}'}{\hat{T}} \right) Y_{3/2} = \left(\frac{\Gamma_\phi}{M_P} \right)^{1/2} \hat{\Sigma} \, \hat{n}_{\rm rad} , \tag{40}
$$

where $\hat{H} \equiv H/\Gamma_\phi$, $\hat{T} \equiv T/(\Gamma_\phi M_P)^{1/2}$, $\hat{n}_{\rm rad} \equiv n_{\rm rad}/(\Gamma_\phi M_P)^{3/2}$ and $\hat{\Sigma} \equiv \langle \sigma_{\rm tot} v_{\rm rel} \rangle M_P^2$. Note that for $v \gg A$, we can approximate Eq. (29) by

$$
\rho(v) \simeq \frac{4}{3} \left(\frac{M_P \Gamma_\phi}{(1 + \bar{w}) v} \right)^2 \equiv (\Gamma_\phi M_P)^2 \hat{\rho}(v) . \tag{41}
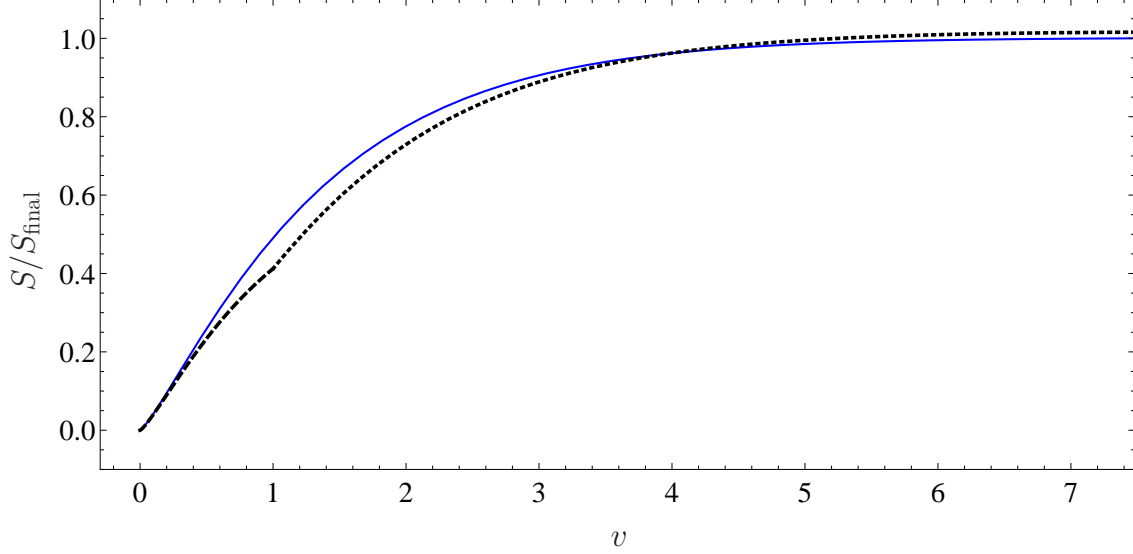$$

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Figure 3: *The entropy $S = sa^3$ as a function of $v = \Gamma_\phi(t - t_{\rm end})$, for the same values of Γ_ϕ and A as in the previous figure. We note that reheating is still incomplete when $v = 1$, and that the net entropy is conserved after reheating ends, for $\delta \ll 1$. The solid blue line is the numerical result, and the dashed (dotted) curve corresponds to the approximate solution (38) for $v < 1$ $(v > 1)$.*

For a constant number of degrees of freedom during reheating, these rescalings allow us to rewrite Eq. (40) as

$$
Y_{3/2}' + 3 \left[\left(\frac{\hat{\rho}_\phi + \hat{\rho}_\gamma}{3} \right)^{1/2} + \frac{\hat{\rho}_\gamma'}{4\hat{\rho}_\gamma} \right] Y_{3/2} = \frac{3}{16\pi} \left(\frac{30\hat{\rho}_\gamma}{\pi^2} \right)^{3/4}
$$
$$
\times\, g^{-3/4} \left(\frac{\Gamma_\phi}{M_P} \right)^{1/2} \sum_{i=1}^{3} c_i\, g_i^2 \left(1 + \frac{m_{\tilde{g}_i}^2}{3m_{3/2}^2} \right) \ln \left(\frac{k_i}{g_i} \right).
$$
$$(42)$$

Disregarding the temperature dependence of the second line, (42) has the formal solution

$$
Y_{3/2}(v) = \frac{3}{16\pi} \left(\frac{30}{\pi^2} \right)^{3/4} \int_0^v \hat{\rho}_\gamma(u)^{3/4} \exp \left[-3 \int_u^v \left\{ \left(\frac{\hat{\rho}(z)}{3} \right)^{1/2} + \frac{\hat{\rho}_\gamma'(z)}{4\hat{\rho}_\gamma(z)} \right\} dz \right] du
$$
$$
\times\, g^{-3/4} \left(\frac{\Gamma_\phi}{M_P} \right)^{1/2} \sum_{i=1}^{3} c_i\, g_i^2 \left(1 + \frac{m_{\tilde{g}_i}^2}{3m_{3/2}^2} \right) \ln \left(\frac{k_i}{g_i} \right).
$$
$$(43)$$

The integrand of this equation provides the contribution to the abundance at the (rescaled) time v of the gravitinos produced at any moment u between 0 and v. For any value of u, the exponential factor accounts for the dilution of those gravitinos due to the inflaton decay, from the times they are produced (u) to the time at which the abundance is evaluated (v). The v-dependence for $v \gg 1$ may be extracted by splitting the integral in the radiation-dominated era, i.e. integrating over u from 0 to ∞, and then subtracting the contribution

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from $v \gg 1$ to ∞. Since the argument of the exponential dilution factor vanishes during radiation domination, the result can be written as

$$\int_0^v \hat{\rho}_\gamma(u)^{3/4} \exp\left[-3\int_u^v \left\{\left(\frac{\hat{\rho}(z)}{3}\right)^{1/2} + \frac{\hat{\rho}'_\gamma(z)}{4\hat{\rho}_\gamma(z)}\right\} dz\right] du$$

$$= \int_0^\infty \hat{\rho}_\gamma(u)^{3/4} \exp\left[-3\int_u^\infty \left\{\left(\frac{\hat{\rho}(z)}{3}\right)^{1/2} + \frac{\hat{\rho}'_\gamma(z)}{4\hat{\rho}_\gamma(z)}\right\} dz\right] du \; - \int_v^\infty \hat{\rho}_\gamma(u)^{3/4} du$$

$$= \int_0^\infty \hat{\rho}_\gamma(u)^{3/4} \exp\left[-3\int_u^\infty \left\{\left(\frac{\hat{\rho}(z)}{3}\right)^{1/2} + \frac{\hat{\rho}'_\gamma(z)}{4\hat{\rho}_\gamma(z)}\right\} dz\right] du \; - 2\left(\frac{3}{4}\right)^{3/4} v^{-1/2}. \quad (44)$$

During reheating, the energy densities may be approximated by (29) and (38). An improved estimate, valid for any $A < v < \infty$, can be constructed if one considers (38) a zeroth-order approximation to ρ_γ. The first-order estimates of the energy densities can then be built by approximating the time-dependent average equation-of-state parameter by

$$\bar{w}(v) \simeq \frac{1}{v}\int_0^v du \frac{\rho_\gamma(u)/3}{\rho_\gamma(u) + \rho_\phi(u)}\bigg|_{w=0} = \frac{1}{3v}\int_0^v \frac{\gamma(\frac{5}{3}, u)}{\gamma(\frac{5}{3}, u) + u^{2/3}e^{-u}}\, du\,, \quad (45)$$

and substituting it into (26), (29) and (32). With this procedure, the first term of (44) is evaluated to be 1.7, which implies that the solution of equation (42) may be approximated for $A \ll 1$ by

$$Y_{3/2}(T) \simeq (0.233 - 0.221\, v^{-1/2})\, g^{-3/4}\left(\frac{\Gamma_\phi}{M_P}\right)^{1/2}\sum_{i=1}^3 c_i\, g_i(T_{\rm reh})^2\left(1 + \frac{m_{\tilde{g}_i}^2}{3m_{3/2}^2}\right)\ln\left(\frac{k_i}{g_i(T_{\rm reh})}\right) \quad (46)$$

after the end of reheating, while the temperature of the thermal bath is still high enough that the number of relativistic degrees of freedom $g = 915/4$.

Figure 4 displays the comparison between (46) and the exact numerical result, showing that they are in agreement within $\lesssim 2\%$ over the entire range $\Gamma_\phi \lesssim 10^{-7}\, M_p$. The final abundance for $T \ll 1\,{\rm MeV}$ is obtained in the limit $v \gg 1$ in (46), including the dilution factor $g(T)/g_{\rm reh}$,

$$Y_{3/2}(T) \simeq 0.233\, \frac{g(T)}{g_{\rm reh}^{7/4}}\left(\frac{\Gamma_\phi}{M_P}\right)^{1/2}\sum_{i=1}^3 c_i\, g_i(T_{\rm reh})^2\left(1 + \frac{m_{\tilde{g}_i}^2}{3m_{3/2}^2}\right)\ln\left(\frac{k_i}{g_i(T_{\rm reh})}\right) \quad (47)$$

$$\simeq 0.00363\left(1 + 0.56\frac{m_{1/2}^2}{m_{3/2}^2}\right)\left(\frac{\Gamma_\phi}{M_p}\right)^{1/2}, \quad (48)$$

where we have assumed in the second line that $g_{\rm reh} = 915/4$, $g(T \ll 1\,{\rm MeV}) = 3.91$, and $T_{\rm reh} \sim 10^{10}\,{\rm GeV}$ for the couplings, neglecting logarithmic corrections. This result can be compared to the full numerical solution of (40), which takes into account the running of the gauge couplings, integrated from the beginning of reheating to $v \gg v_{\rm reh}$, deep into the
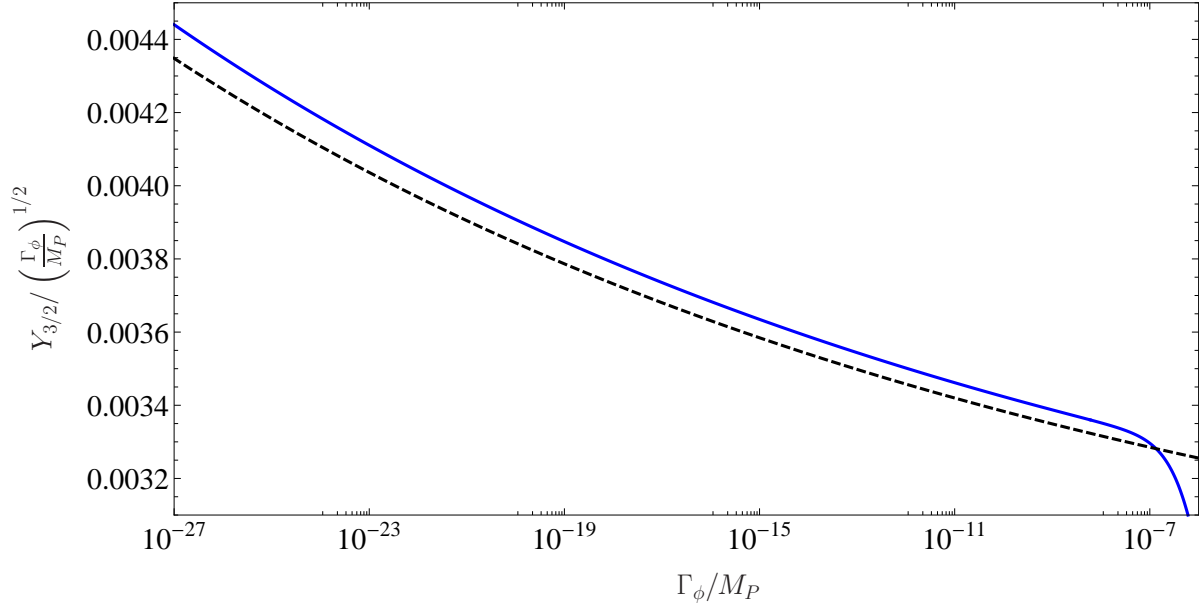
Figure 4: *The final gravitino yield $Y_{3/2}(T \ll 1\,\mathrm{MeV})$ as a function of the inflaton decay rate, assuming instantaneous thermalization during inflaton decay, for the $m_{1/2}$-independent transversal components. The normalization is chosen to emphasize the dependence of $Y_{3/2}$ on the running of the coupling constants. The blue continuous line corresponds to the numerical solution obtained from equations (21)-(23). The black dashed line shows the approximation (46). Here we assume the MSSM value $g = 915/4$ for the number of degrees of freedom during reheating.*

radiation-dominated era, where the gravitino yield asymptotes to its final value. Accounting for the dilution factor, a phenomenological fit to this numerical result gives

$$Y_{3/2}(T) = 0.00360 \left(\frac{\Gamma_\phi}{M_P}\right)^{1/2} \tag{49}$$

for the $m_{1/2}$-independent part in the range $10^{-20} \leq \Gamma_\phi/M_P \leq 10^{-8}$.

We are now in a position to compare our exact result (49) with the naive approximation made previously in Section 2. In Fig. 5 we show the evolution of the gravitino abundance as a function of time (as parametrized by v). The solid curve shows the exact solution, which asymptotes to $Y_{3/2}/(\Gamma_\phi/M_P)^{1/2} \simeq 0.203$. Since $T(v = 10^4) \sim 10^{10}\,\mathrm{GeV}$ for the decay rate considered, this result would still need to be multiplied by the dilution factor $g(T)/g_{\mathrm{reh}} \simeq 0.0171$ in order to obtain the final abundance, $Y_{3/2}(T \ll T_{\mathrm{reh}}) \simeq 3.44 \times 10^{-9}$, in very good agreement with (48) and (49), which give $Y_{3/2} \simeq 3.63 \times 10^{-9}$ and 3.60×10^{-9}, respectively (in comparing with these relations, recall that $\Gamma_\phi = 10^{-12} M_P$ is assumed in the example shown in the Figure). The blue curve tracks the abundance at $v < 1$, and the dilution of that production is tracked by the dashed blue curve at $v > 1$. Production (and dilution) at $v > 1$ is shown by the dashed red curve, and the sum of the two dashed curves

gives the solid black curve. The result (49) demonstrates that the gravitino abundance is sensitive primarily to the final reheating temperature, after the production of entropy has ceased, rather than to the maximum temperature of the Universe seen in Fig. 2, as originally pointed out in [19].
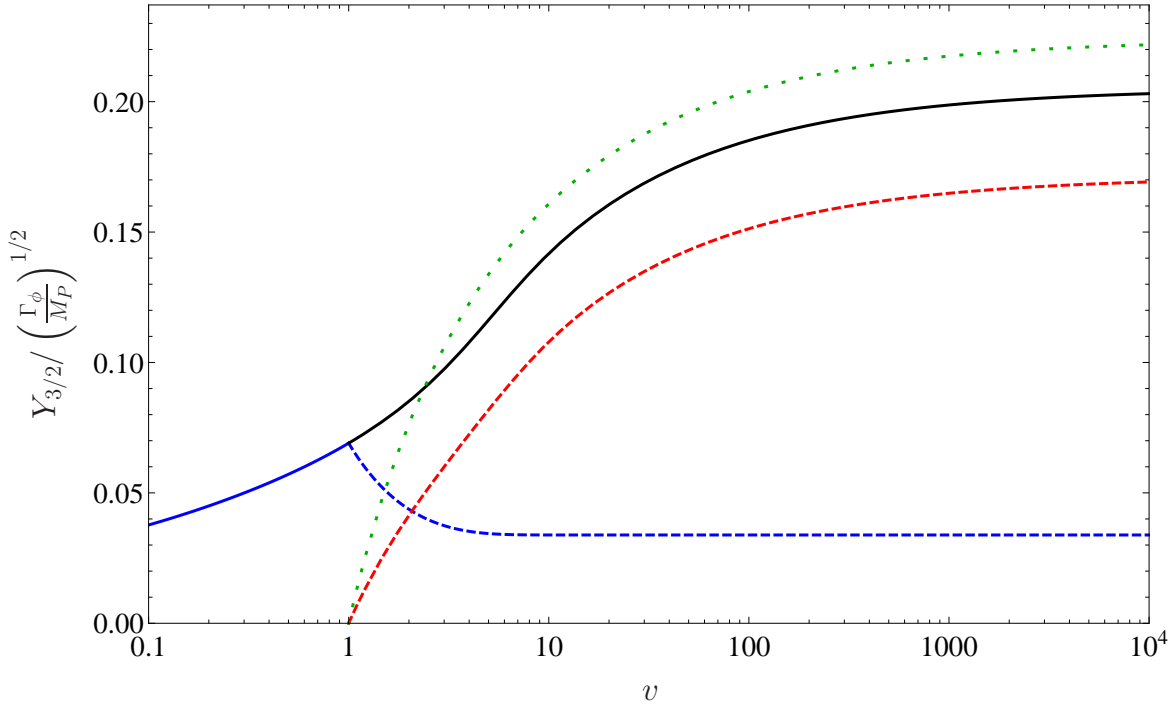

Figure 5: *The evolution of the gravitino abundance, as a function of $v = \Gamma_\phi(t - t_{\text{end}})$ during and after reheating, for the decay rate $\Gamma_\phi = 10^{-12}M_P$. The solid curve corresponds to the numerical solution of (40), assuming $g_{\text{reh}} = 915/4 =$const. (i.e. no dilution due to the factor $g(T)/g_{\text{reh}}$ has been included), and neglecting $m_{1/2}$-dependent terms in the collision term. The blue curve shows the evolution of the yield produced only at $v < 1$; the dilution of that yield for $v > 1$ is shown in the blue dashed curve. The red dashed curve tracks the abundance produced at $v > 1$, and the sum of the dashed curves is shown as the black solid curve. The dotted green curve demonstrates the evolution of the gravitino yield assuming instantaneous decay and thermalization at $v = 1$.*

The exact result should be compared with the green dotted curve that assumes instantaneous decay at $v = 1$, namely $c = 1$ in eqs. (18) and (19). Impressively, at large v these results lie within 10% of each other [3]. As noted earlier, another common choice for the instantaneous decay is $c = 2/3$ (namely $\Gamma_\phi = H$), and in this case we would have an asymptotic yield of 0.272, which would correspond to $Y_{3/2}(T \ll T_{\text{reh}}) \simeq 4.65 \times 10^{-9}$. The ratio of the naive (19) to the exact (49) result is shown in Fig. 6 as a function of $c = \Gamma_\phi t_{\text{inst. reh.}}$. We

---

[3]The difference between the asymptotic value for the yield of 0.222 for $c = 1$ shown in Fig. 5 and the expected value of 0.233 from (19) is due to the logarithmic correction in (16).

see that the instantaneous approximation yields the correct result when $c \simeq 1.2$ [24].
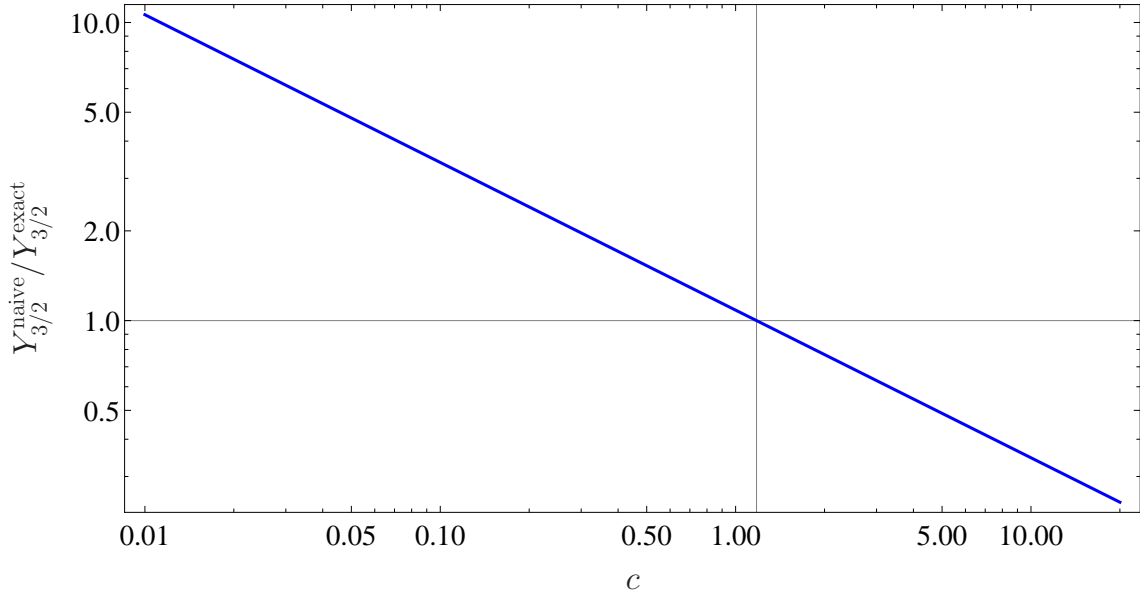

Figure 6: *The ratio of the final gravitino yield $Y_{3/2}^{\text{naive}}$, assuming instantaneous decay and thermalization at $v = c$, to the exact yield. The results agree for $c \simeq 1.2$.*

So far we have assumed that thermalization occurs instantaneously upon the end of inflation. However, since the initial distribution of the decay products of the inflaton is not thermal, the thermalization of the relativistic plasma will not be completed until the interactions of the constituent particles are sufficient to create a thermal distribution. In the next Section we study in more detail the approach to kinetic and chemical equilibrium, finding that, for a small decay rate $\Gamma_\phi \ll m$, number-conserving and number-changing processes at small scattering angles would efficiently thermalize the plasma well before the end of reheating, $v_{\text{th}} \ll v_{\text{reh}}$. It is in any case important to quantify the dependence of the final gravitino abundance on the thermalization rate. The comparison between (19), with $c = \text{O}(1)$, and (49) suggests that the bulk of the relic gravitino density is produced at $v \sim 1$, and therefore $Y_{3/2}$ may only have a weak dependence on v_{th}. Nonetheless, a non-negligible fraction of the total gravitino abundance may still be produced at $v \ll 1$, since at early times $T > T_{\text{reh}}$.

We assume here that the distribution of the inflaton decay products is non-thermal before v_{th}, and we disregard gravitino production at this early stage, thus obtaining a lower bound on the total amount of produced gravitinos. Without entering in the details of thermalization (which are discussed in the next Section), here we treat v_{th} as a free parameter. The goal of this analysis is to understand how much the final gravitino abundance is dependent on this parameter and, ultimately, how much our final result (49) is affected by possible uncertainties on the thermalization processes. The results summarized in Fig. 7 show that the final abundance (49) is extremely robust. This result holds with very good accuracy provided

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that $v_{\rm th} \lesssim 0.1$. Essentially, only the gravitinos produced when $v \gtrsim 0.1$ contribute to the final abundance, as the gravitinos produced at earlier stages are diluted away. It is possible that thermalization has not taken place by the time $v_{\rm max} \ll 1$ given in Eq. (35), so that it is possible that the maximum temperature (36), is not reached (we discuss this in detail in Subsection 4.2). However, this does not affect the validity of (49). For the result (49) to be valid, we only need to assume that the inflaton decay products are thermalized by $v \simeq 0.1$. In the next Section we verify that this is indeed the case.
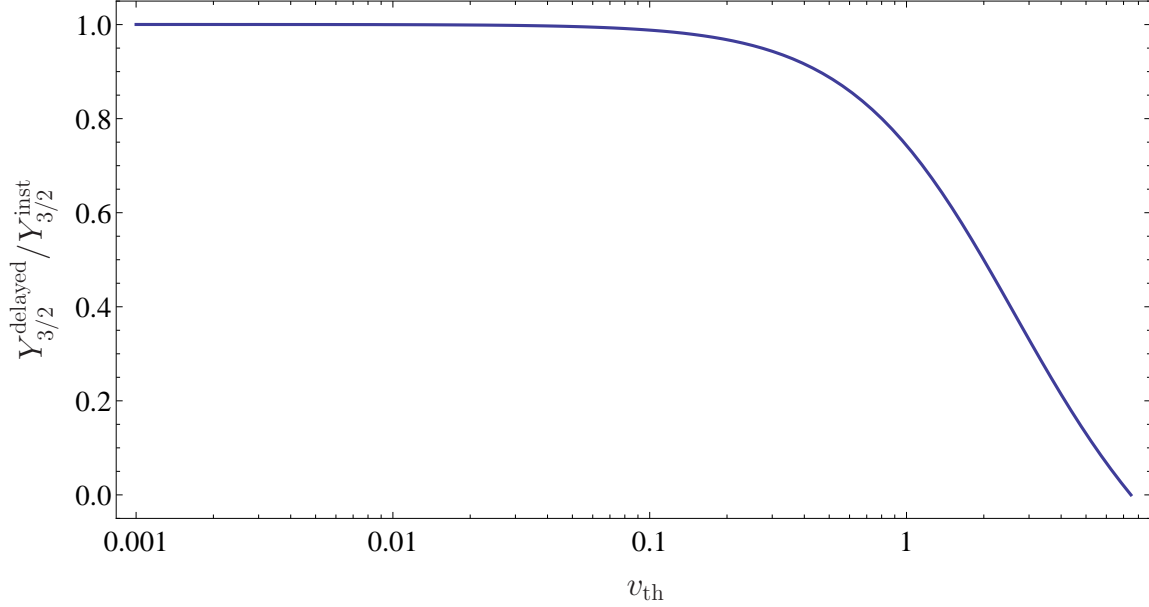

Figure 7: *The ratio of the gravitino yield at the end of reheating $Y_{3/2}(T_{\rm reh})$ assuming delayed thermalization at $v_{\rm th}$, to the yield assuming instantaneous thermalization, as a function of the thermalization delay $v_{\rm th}$. For $Y_{3/2}^{\rm delayed}$, production at $v < v_{\rm th}$ is ignored. The suppression of the final abundance is negligible unless $v_{\rm th} \gtrsim 0.1$*

4   The Thermalization Process

In the previous sections we have assumed that the inflaton decay products thermalize instantaneously. We now discuss this approximation, studying the relevant physical processes and the timescale for the thermalization. In the limit in which the created particles are not interacting, their number density will be given by

$$n_p(t) \simeq \frac{\rho_{\rm end}}{m} \left(1 - e^{-\Gamma_\phi(t-t_{\rm end})}\right) \times \left(\frac{a(t)}{a_{\rm end}}\right)^{-3}, \tag{50}$$

up to an order one factor that we disregard here. This number density should be compared to the thermal one, given by

$$n_{\text{th}}(t) = \frac{\zeta(3)}{\pi^2} \tilde{g}(T) \left(\frac{30\rho_\gamma(t)}{\pi^2 g(T)} \right)^{3/4} , \tag{51}$$

where $\tilde{g} \equiv \sum_B g_B + \frac{3}{4} \sum_F g_F$ denotes the effective number of relativistic degrees of freedom contributing to the number density. From this result and from (38) we obtain

$$
\begin{aligned}
\frac{n_{\text{th}}}{n_p} &\simeq 0.28 \frac{\tilde{g}(T)}{g(T)^{3/4}} \frac{m}{\rho_{\text{end}}^{1/4}} \frac{e^{\frac{3A}{4}} \left(\boldsymbol{\gamma}\left(\frac{5}{3}, A+v\right) - \boldsymbol{\gamma}\left(\frac{5}{3}, A\right) \right)^{3/4}}{\sqrt{A} \left(1 - e^{-v}\right)} \\
&\simeq 0.26 \frac{\tilde{g}(T)}{g(T)^{3/4}} \frac{m}{(\Gamma_\phi M_P)^{1/2}} \frac{\boldsymbol{\gamma}\left(\frac{5}{3}, v\right)^{3/4}}{1 - e^{-v}} ,
\end{aligned}
\tag{52}
$$

where the first expression assumes $v \lesssim 1$, while the second expression assumes $A \ll v \lesssim 1$.

We can simplify this expression by noting that the last factor in (52) interpolates between $0.68 \, v^{1/4}$ at $v \ll 1$ and 0.69 at $v = 1$. We then use $g(T) = 915/4$, $\tilde{g}(T) = 427/2$, and we parametrize the inflaton decay rate as $\Gamma_\phi = m|y|^2/8\pi$. We obtain

$$\frac{n_{\text{th}}}{n_p} \simeq 3.2 \, v^{1/4} \frac{\sqrt{m}}{|y| \sqrt{M_P}} \quad , \quad A \ll v \lesssim 1 . \tag{53}$$

Assuming $m \simeq 10^{-5} M_P$, we thus see that, if they do not interact, the decay products are in a regime of 'under-occupation' with respect to the thermal case (namely, $n_{\text{th}}/n_p > 1$) for $|y| \lesssim \mathcal{O}\left(10^{-2}\right) v^{1/4}$; since $v < v_{\text{max}} \sim |y|^2/8\pi$, this is equivalent to $|y| \lesssim \mathcal{O}\left(10^{-5}\right)$ for $v > v_{\text{max}}$. This is typically the case, for example, in no-scale models [6], when the bulk of the inflaton quanta decay. We conclude that processes that increase the number of quanta need to be effective for thermalization to take place. We now consider some mechanisms for thermalization of the inflaton decay products.

4.1 Large-Angle Scattering

As we discussed in the previous section, number-increasing processes are necessary to thermalize the relativistic decay products of the inflaton. However, elastic scatterings are also of interest, since they can bring the plasma into kinetic equilibrium.

We start by considering the $2 \leftrightarrow 2$ elastic scatterings among the decay products. The cross section decreases as the square of the transfer momentum, which, at large angle, is of the order of the momentum of the incoming particles, $\sigma \sim \alpha^2/p^2$, where α denotes the coupling strength. Therefore the rate is greatest for the particles produced at the earliest time, once their energy redshifts from the initial $\mathcal{O}(m)$ energy to

$$p_{\text{redshifted}} \simeq \frac{m}{2} \frac{a_{\text{end}}}{a(t)} \simeq \frac{m^{1/3}}{2 \left(t - t_{\text{end}}\right)^{2/3}} , \qquad t - t_{\text{end}} \gg m^{-1} . \tag{54}$$

One could then imagine that the redshifted inflaton decay products can thermalize among themselves, and that more energetic particles, produced by later inflaton decay, can in turn thermalize by interacting with this soft, thermalized tail after it has reached equilibrium [42].

However, one needs to take into account that, at any moment during the decay of the inflaton, hard particles with $p \sim m$ (namely, those produced around that moment) are more abundant than the previously-produced redshifted decay products. At any given time, the spectrum of the decay products can be found approximately by solving the Boltzmann transport equation in the absence of interactions:

$$\frac{\partial f_p}{\partial t} - H p \frac{\partial f_p}{\partial p} \simeq 0 \,, \tag{55}$$

where $f_p(t)$ denotes the distribution density of the decay products, $n_p = \frac{g_p}{(2\pi)^3} \int d^3 p \, f_p$. When thermalization is complete this distribution density depends on time only through the temperature, $f_p \sim e^{-p/T}$. The solution of equation (55) that matches the early time limit of (50) using (32), $A \ll v \ll 1$, namely

$$n_p \simeq \frac{4 \, \Gamma_\phi^2 \, M_p^2}{3 \, m \, v} \,, \tag{56}$$

is

$$f_p(t) = \frac{8\sqrt{2}\pi^2}{g_p} \frac{\Gamma_\phi^2 M_P^2}{m^4} \left(\frac{m}{p}\right)^{3/2} v^{-1} \,, \tag{57}$$

which implies a spectrum of the form $\frac{dn}{dp} \sim \frac{\Gamma_\phi^2 M_P^2 p^{1/2}}{m^{5/2} v}$. It follows that the large-angle scattering cross section is IR dominated:

$$\frac{\Gamma_{\text{elas}}}{H} = \frac{1}{H} \int \sigma \, dn \simeq \frac{1}{H} \int_{p_{\text{redshifted}}}^{\frac{m}{2}} dp \, \frac{dn}{dp} \frac{\alpha^2}{p^2} \simeq \alpha^2 \left(\frac{M_P}{m}\right)^2 \left(\frac{\Gamma_\phi}{m}\right)^{2/3} v^{1/3} \,. \tag{58}$$

Assuming $m \simeq 10^{-5} M_P$ and scattering mediated by the strong interaction, the ratio (58) shows that $2 \leftrightarrow 2$ processes can achieve kinetic equilibrium at $v < 1$, if the decay rate is $\Gamma_\phi \gtrsim 0.05 \, m^3/M_P^2$. However, in some no-scale models the inflaton decay rate lies below this bound [39].

When the rate of the $2 \leftrightarrow 2$ scatterings is too small, processes that increase particle number must be effective, not only to increase the overall number density to the thermal one, but also to produce soft particles efficiently, if one hopes to thermalize the plasma before the end of reheating. At large angles, the rate for $2 \to 3$ splitting, which is the most efficient inelastic process, may be approximated by $\sigma \sim \alpha^3/p^2$ [41]. We can then mirror the analysis leading to (58), but with a rate further suppressed by an extra factor of α. Under the same assumptions, thermalization would occur before $v = 1$ only if $\Gamma_\phi \gtrsim 4.4 \, m^3/M_P^2$. We conclude that, for a decay rate of the inflaton arising from Planck-suppressed interactions, thermalization before the end of reheating generically does not occur via large-angle scattering processes, since (1) not enough particles can be created to populate the thermal bath, and (2) the spectrum of the existing decay products remains too hard. However, as we discuss in the following subsection, these two shortcomings can be cured by considering the small-angle scattering of the inflaton decay products.

4.2 Small-Angle Scattering

At small angles, the momentum transfer vanishes, and so the cross section is in general infrared divergent in the vacuum. Due to the increase of the rate, a more careful kinetic-theory approach than the one used in the previous subsection has to be adopted [43]. The Boltzmann equation controls the evolution of the (uncorrelated) one-particle distribution functions under the assumption that the relaxation time is much larger than the collision time, so there is no interference between successive scatterings. However, this assumption is not valid in gauge theories, as the mean free time between small-angle scatterings in a thermal bath can be shown to be of the same order as the time for formation of a bremsstrahlung gauge boson, $\tau \sim 1/g^2 T$ [44]. This results in a suppression of the bremsstrahlung rate, since now several successive small-angle collisions become virtually indistinguishable from a single collision (the Landau-Pomeranchuk-Migdal (LPM) effect) [45]. Nevertheless, the introduction of the full Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations for a system of interacting particles can be avoided by setting up an effective Boltzmann equation, which can be written schematically as [43]

$$\frac{\partial f_p}{\partial t} - Hp\frac{\partial f_p}{\partial p} = \left|\ \vcenter{\hbox{}}\ \right|^2 + \left|\ \vcenter{\hbox{}}\ \right|^2 + \cdots$$

$$\equiv -\mathcal{C}^{2\leftrightarrow2}[f_p] - \mathcal{C}^{``1\leftrightarrow2"}[f_p] + \cdots , \tag{59}$$

where the elastic and inelastic collision terms can be written as

$$\mathcal{C}^{2\leftrightarrow2}[f_p] = \frac{1}{2}\int \frac{d^3k\,d^3p'\,d^3k'}{(2\pi)^9}\frac{|\mathcal{M}(p,k;p',k')|^2}{(2p_0)(2k_0)(2p_0')(2k_0')}(2\pi)^4\delta^{(4)}(p+k-p'-k')$$

$$\times \left\{f_p f_k[1 \pm f_{p'}][1 \pm f_{k'}] - f_{p'}f_{k'}[1 \pm f_p][1 \pm f_k]\right\}, \tag{60}$$

$$\mathcal{C}^{``1\leftrightarrow2"}[f_p] = \int \frac{dk}{(2\pi)8p^2}\,\gamma(p;k)\left\{f_p[1 \pm f_{p-k}][1 + f_k] - f_{p-k}f_k[1 \pm f_p]\right\}. \tag{61}$$

The leading-order processes are illustrated on the right-hand side of (59): elastic $2 \leftrightarrow 2$ scattering, and near-collinear $1 + N \leftrightarrow 2 + N$ particle-splitting processes. The LPM effect is taken into account by using an effective splitting rate that sums up all interferences in the bremsstrahlung process; the function $\gamma(p;k)$ corresponds to the near-collinear amplitude including the LPM effect, the phase-space integration and all necessary symmetry factors.

Mirroring the steps in the previous section, we discuss first the consequences of this for the elastic term. In particular, we consider as an example gluon-gluon scattering: $g\,(\mathbf{p})+g\,(\mathbf{k}) \to g\,(\mathbf{p}') + g\,(\mathbf{k}')$. In the vacuum case one obtains a IR-divergent rescaled squared amplitude:

$$|M(p,k;p',k')|^2 = \frac{|\mathcal{M}(p,k;p',k')|^2}{(2p_0)(2k_0)(2p_0')(2k_0')} \sim \frac{\alpha^2}{(q_\perp^2)^2}, \tag{62}$$

in the limit of small momentum exchange $q_\perp = |\mathbf{p} - \mathbf{p}'|_\perp$. Including the effect of the successive scatterings in the plasma effectively results [46] in replacing (62) by:

$$|M|^2 \sim \frac{\alpha^2}{q_\perp^2 (q_\perp^2 + m_s^2)} \, , \tag{63}$$

where m_s denotes the screening scale, $m_s^2 \sim \alpha \int d^3p \, f_p/p$, which scales as αT^2 once thermal equilibrium has been established. With the elastic scattering rate given by

$$\Gamma_{\text{elas}} \sim \int d^3p \, d^3p' \, d^3k' \, |M|^2 \, \delta^{(4)}(p + k - p' - k') \, f_p [1 \pm f_{p'}][1 \pm f_{k'}] \tag{64}$$

one then finds that, for $q_\perp \ll |\mathbf{p}|$ [46],

$$\Gamma_{\text{elas}} \sim \int d^2q_\perp \frac{\alpha^2}{q_\perp^2 (q_\perp^2 + m_s^2)} \int d^3p \, f_p)[1 \pm f_p][1 \pm f_k] \tag{65}$$

$$\sim \frac{\alpha^2}{m_s^2} \int d^3p \, f_p [1 \pm f_p][1 \pm f_k] \, . \tag{66}$$

Even assuming that $1 \pm f_p \sim \mathcal{O}(1)$, we find a rate $\Gamma_{\text{elas}} \sim \alpha \, p \sim \alpha \, m$, which is much larger than $H \simeq 2\Gamma_\phi/3v$ long before the end of reheating, for any decay rate $\Gamma_\phi/m \ll \alpha$. As we see below, number-increasing processes also become effective, leading to $f_p \gg 1$ (and thus to a even greater elastic cross section) in the bosonic case. Even though elastic processes do not increase the number of quanta, they can effectively redistribute energies towards kinetic equilibrium.

Let us now discuss number-increasing processes. We focus the discussion on the production of "hard" gauge bosons, namely of a gauge boson that carries a non-negligible fraction of the energy E of the primary particle. If such processes are efficient, their combined effect thermalizes the decay products by increasing the total number of quanta (pushing the ratio (52) towards unity) and by decreasing the average energy per quantum. Namely, a hard primary loses energy by emitting a gauge boson with comparable, but softer, momentum E, which, during a time comparable to its creation time t_γ, splits in turn into two gauge bosons with comparable momenta (the so-called *hard branching*). The products of this branching quickly cascade further, giving their energy to the thermal bath [47]. As discussed in [48, 49], the typical time t_γ needed for a near-collinear emission of a hard gauge boson can be estimated as

$$t_\gamma \sim \sqrt{\frac{\tau E}{q_\perp^2}} \, , \tag{67}$$

where τ is the mean time between collisions between the primary and the plasma; this expression is valid for $t_\gamma \lesssim E/q_\perp^2$ [46] [4]. We now specialize to the hard particles with $E \sim m$ produced in inflaton decay with distribution function (57). In the non-equilibrium case, the mean free time between small-angle scatterings, with momentum transfer of order $q_\perp \sim m_s$,

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[4]For the creation of a hard boson, this condition is equivalent to $\Gamma_\phi \lesssim (\frac{m^2}{M_P}) v^{1/2} \simeq 10^5 (\frac{m^3}{M_P^2}) v^{1/2}$.

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scales as [48] $\tau^{-1} \sim \frac{\alpha^2}{m_s^2} n_p$. To estimate when the number-increasing scatterings become effective, we use the expression (56) for the number density n_i of the hard decay products, and

$$m_s^2 \sim \alpha \frac{\Gamma_\phi^2 M_P^2}{m^2} v^{-1}, \qquad (68)$$

for the screening mass scale. This results in the following time-dependent production rate for a hard photon or gluon:

$$\frac{\Gamma_{\gamma,\,\mathrm{hard}}}{H} \sim \alpha \left(\frac{M_P}{m} \right) v^{1/2}, \qquad (69)$$

which is $\gg 1$ much earlier than the end of reheating. Therefore, complete thermalization can take place long before the start of the radiation-dominated era. It is worth noting that the hard gauge boson emission time-scale (and therefore the thermalization time-scale) is in general $v_{\mathrm{hard}} > v_{\mathrm{max}}$, which implies that the maximum temperature of the relativistic plasma during the reheating era is model-dependent and smaller than the maximum temperature T_{max} (36) assuming instantaneous thermalization.

A more detailed account of the thermalization process, tracking the phase-space distributions of the hard and soft sectors, can be found in [50, 51]. The thermalization time-scale found therein is parametrically of the same order of magnitude or smaller than the scale of hard-boson emission v_{hard} discussed above for a Planck-suppressed decay rate.

5   Non-Thermal Gravitino Production

So far we have focused on the production of gravitinos by scattering processes in the thermally-equilibrated plasma. In this Section we study two different mechanisms of non-thermal gravitino production during reheating. Specifically, in Subsection 5.1 we study the gravitinos produced by the hard inflaton decay products, before they thermalize, and in Subsection 5.2 we study the amount of gravitinos produced by inflaton decays.

5.1   Gravitino production from hard inflaton decay products

In the earliest stages of reheating, before thermalization takes place, some gravitinos would have been produced by the scattering of the hard decay products with momenta $p \sim m$ and distribution function (57). If the number density of hard primaries were sufficiently large, the production rate would be enhanced relative to the thermal one. In this subsection we study whether this could increase significantly the final yield $Y_{3/2}$.

For definiteness, we consider the specific scenario in which the inflaton field decays predominantly into gauge bosons, $\phi \to gg$: such a scenario is possible in no-scale supergravity models of inflation with a non-trivial gauge kinetic function [6, 40, 52]. We can then compute the non-thermal gravitino production via the channel $g^a(\mathbf{k}_1) + g^b(\mathbf{k}_2) \to \psi_{3/2}(\mathbf{p}_1) + \tilde{g}^c(\mathbf{p}_2)$,

with scattering amplitude [20] [5]

$$|\mathcal{M}|^2 = \frac{4g^2}{M_P^2}|f^{abc}|^2 \left(1 + \frac{m_{1/2}^2}{3m_{3/2}^2}\right)\left(s + 2t + 2\frac{t^2}{s}\right),\tag{70}$$

where t, s are Mandelstam variables. The Boltzmann transport equation for the gravitino distribution function is then given by

$$\frac{\partial f_{p_1}}{\partial t} - Hp_1 \frac{\partial f_{p_1}}{\partial p_1} = \frac{1}{2(2p_1)}\int \frac{d^3\mathbf{k}_1}{(2\pi)^3 2k_1}\frac{d^3\mathbf{k}_2}{(2\pi)^3 2k_2}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2p_2}(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)|\mathcal{M}|^2$$
$$\times \left\{ f_{k_1}f_{k_2}[1 - f_{p_2}][1 - f_{p_1}] - f_{p_2}f_{p_1}[1 + f_{k_1}][1 + f_{k_2}]\right\}.\tag{71}$$

The distribution functions $f_{k_{1,2}}$ for the gauge bosons may be approximated by (57); in the limit $\Gamma_\phi/m \ll 1$, they are $f_k < 1$. In this approximation, we integrate the creation term of the Boltzmann equation with respect to the gravitino momentum, obtaining

$$\frac{dn_{3/2}}{dt} + 3Hn_{3/2} \simeq \frac{g^2|f^{abc}|^2\Gamma_\phi^2 M_P^2}{\pi^3(mt)^2}\left(1 + \frac{m_{1/2}^2}{3m_{3/2}^2}\right)$$
$$\times \int dx\,dy\,dz\ G(x,y,z)x^{-3/2}(y + z - x)^{-3/2}\,\Omega(x,y,z;t),\tag{72}$$

where the function $G(x, y, z)$ is defined as

$$G(x,y,z) \equiv \frac{(x + z - |x - z|)}{(y + z)\,|x - z|}\left[(x - z)^2\left(z(2x - z) - y^2 - 2yz\right)\right.$$
$$\left. + |x - z|\left(z(y + z)^2 + 2x^2 z + \left(y^2 - 2yz - z^2\right)x\right)\right],\tag{73}$$

and $\Omega(x, y, z; t)$ parametrizes the time-dependent integration limits,

$$\Omega(x,y,z;t) = \theta(y)\,\theta(z)\,\theta\left(\tfrac{1}{2} - x\right)\,\theta\left(\tfrac{1}{2} - (y + z - x)\right)$$
$$\times \theta\left(x - \tfrac{b}{2}(mt)^{-2/3}\right)\theta\left((y + z - x) - \tfrac{b}{2}(mt)^{-2/3}\right),\tag{74}$$

where $b = \left(\frac{3}{4}\frac{\rho_{\text{end}}}{m^2 M_P^2}\right)^{-1/3}$. Due to the complicated time dependence of the collision term in (72), it is useful to consider a particular time in order to compare it to the thermal collision term. The right-hand side of (72) can be written as

$$\mathcal{C}_{\text{non-thermal}} \equiv \frac{g^2|f^{abc}|^2\Gamma_\phi^2 M_P^2}{\pi^3}\left(1 + \frac{m_{1/2}^2}{3m_{3/2}^2}\right)F(t),\tag{75}$$

where the function

$$F(t) \equiv (mt)^{-2}\int dx\,dy\,dz\ G(x,y,z)x^{-3/2}(y + z - x)^{-3/2}\,\Omega(x,y,z;t)\tag{76}$$

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[5]The corresponding cross section for this process is infrared-finite, which allows a simple analysis.

is maximized at $mt \simeq 5.62$ with value $F \simeq 3.1 \times 10^{-3}$, after which it is monotonically decreasing. Evaluating then at $mt \simeq 5.62$, close to the maximum temperature during reheating $t_{\max} \sim m$ (35), we find the following relation between the thermal collision term $\mathcal{C}_{\text{thermal}}$, given by the right-hand side of (3), and the non-thermal collision term $\mathcal{C}_{\text{non-thermal}}$ defined by (75),

$$\left.\frac{\mathcal{C}_{\text{non-thermal}}}{\mathcal{C}_{\text{thermal}}}\right|_{mt\simeq 5.62} \simeq \frac{10^{-4}\, g_3^2 |f^{abc}|^2 \Gamma_\phi^2 M_P^2}{0.007\, c_3 g_3^2 \ln(k_3/g_3) T_{\max}^6 / M_P^2} \sim \frac{160}{\ln(k_3/g_3)} \left(\frac{\Gamma_\phi}{m}\right)^{1/2} \left(\frac{M_P}{m}\right), \quad (77)$$

where we have considered only the dominant $SU(3)$ component and disregarded $m_{1/2}$-dependent terms for simplicity. Equation (77) indicates that, for an inflaton decay rate $\Gamma_\phi \gtrsim 10^{-4} (m/M_P)^2 m \sim 10^{-14} m$, the instantaneous rate for direct production from the hard decay products is larger than the thermal production rate. However, this is true only before thermalization is complete, i.e., $v < v_{\text{th}}$, after which the distribution functions have their thermal forms. After thermal equilibrium is achieved, the non-thermally-produced abundance is rapidly diluted by the growing entropy density, resulting in a final abundance that is virtually indistinguishable from the abundance that would be produced in the instantaneous-thermalization case. We have verified these results by tracking numerically the gravitino yield for a decay rate $\Gamma_\phi = 10^{-10} m$ including non-thermal production.

5.2 Gravitino production by inflaton decays

Gravitinos may also be created by direct inflaton decay. Let us denote by $B_{3/2}$ the branching ratio of the decay to gravitinos; we implicitly assume that the number of gravitinos produced per inflaton quanta is also factored into $B_{3/2}$. Given that the number of quanta that have decayed at a time t is given by (50), this non-thermally produced gravitino population during reheating evolves as

$$n_{3/2} = B_{3/2} \frac{\rho_\phi}{m} (e^v - 1). \tag{78}$$

Since thermalization occurs rapidly, with $v_{\text{th}} \ll 1$, we can write the ratio $n_{3/2}/n_{\text{rad}}$, with the instantaneous temperature given by (20), as

$$\begin{aligned} Y_{3/2,\text{from decay}}(v) &= \frac{\pi^2 B_{3/2}}{\zeta(3)m} \left(\frac{g\pi^2}{30}\right)^{3/4} \frac{\rho_\phi}{\rho_\gamma^{3/4}} (e^v - 1) \\ &= \frac{\pi^2 B_{3/2}}{\zeta(3)m} \left(\frac{g\pi^2}{30}\right)^{3/4} \rho^{1/4} \Omega_\gamma^{-3/4} (1 - \Omega_\gamma)(e^v - 1). \end{aligned} \tag{79}$$

The right-hand side of (79) can be evaluated after reheating has ended by noting that, during the radiation-dominated era, $\rho \approx \rho_\gamma \approx \frac{3}{4}(\Gamma_\phi M_P)^2 v^{-2}$, and combining (26) and (32) with $w = 1/3$ for the late-time solution for ρ_ϕ,

$$1 - \Omega_\gamma = \frac{\rho_\phi}{\rho} \simeq \frac{\frac{4}{3}(\Gamma_\phi M_P)^2 v^{-3/2} e^{-v}}{\frac{3}{4}(\Gamma_\phi M_P)^2 v^{-2}} = \frac{16}{9} v^{1/2} e^{-v}. \tag{80}$$

After inclusion of the dilution factor $g(T)/g_{\text{reh}}$, this leads to

$$Y_{3/2,\text{from decay}}(T \ll T_{\text{reh}}) = \frac{\pi^2 B_{3/2}}{\zeta(3)} \left(\frac{4}{3}\right)^{7/4} \frac{g(T)}{g_{\text{reh}}} \left(\frac{g_{\text{reh}}\pi^2}{30}\right)^{3/4} \frac{\sqrt{\Gamma_\phi M_P}}{m}(1 - e^{-v}) \qquad (81)$$

$$\simeq 5.9\, B_{3/2} \frac{\sqrt{\Gamma_\phi M_P}}{m}\,. \qquad (82)$$

Comparing (82) to the thermally-produced yield (49), one finds the ratio

$$\frac{Y_{3/2,\text{from decay}}}{Y_{3/2,\text{thermal}}} \simeq 1.6 \times 10^3 B_{3/2} \left(\frac{M_P}{m}\right)\,. \qquad (83)$$

Hence the direct decay result would dominate if $B_{3/2} \gtrsim 10^{-8}$ for $m \simeq 10^{-5} M_P$.

This branching ratio is model-dependent, and we focus our attention in the following on no-scale models. In [39] we studied the decay of an untwisted sneutrino inflaton with Starobinsky potential, finding that the dominant decay rate in the presence of the Yukawa-like term $W \supset y_\nu H_u L\phi$ is that into the matter-Higgs channels

$$\Gamma(\phi \to H_u^0 \tilde{\nu}, H_u^+ \tilde{f}_L) + \Gamma(\phi \to \tilde{H}_u^0 \nu, \tilde{H}_u^+ f_L) = m\frac{|y_\nu|^2}{8\pi}\,, \qquad (84)$$

while the decay into gravitinos occurs with rate

$$\Gamma(\phi \to \psi_{3/2}\nu) = v^2 \sin^2\beta\, m\frac{|y_\nu|^2}{32\pi M_P^2}\,, \qquad (85)$$

where in this context v denotes the Higgs vev. The decay of the inflaton to a gravitino and an inflatino may also be possible if it is kinematically allowed, with a rate

$$\Gamma(\phi \to \psi_{3/2}\tilde{\phi}) \sim \left(\frac{m_{3/2}}{m}\right)^2 \frac{17m^3}{48\pi M_P^2}\,. \qquad (86)$$

The factor $(m_{3/2}/m)^2$ represents a suppression due to the near degeneracy of the inflaton and inflatino [33]. The corresponding branching ratios are negligible: $B_{3/2} \sim (10^{-33}, 10^{-27}|y_\nu|^{-2})$ for $m_{3/2} \sim 100\,\text{TeV}$.

An additional channel for inflaton decay into gravitinos may arise from a superpotential term coupling the inflaton ϕ to the volume modulus T responsible for supersymmetry breaking, of the form

$$W \supset \zeta(T - 1/2)^2\phi\,. \qquad (87)$$

Such terms do not spoil the inflationary potential, and may lead to a large rate for decays into gravitinos:

$$\Gamma(\phi \to \psi_{3/2}\psi_{3/2}) = m\frac{|\zeta|^2}{72\pi}\,. \qquad (88)$$

Assuming that the dominant channels correspond to the matter-Higgs decays (84), and accounting for the two gravitinos that are produced in each decay, we have

$$B_{3/2} = \frac{2|\zeta|^2}{9\,|y_\nu|^2}\,, \qquad (89)$$

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which can be sizeable.

In the case of a volume modulus inflaton T, the decay rates were also evaluated in [39]. The dominant channels correspond to the three-body decays

$$\Gamma(T \to H_u^0 t_L \bar{t}_R, \; \tilde{t}_L \tilde{H}_u^0 \bar{t}_R, \; \bar{\tilde{t}}_R t_L \tilde{H}_u^0) = (2n_t + n_H - 3)^2 \frac{|y_t|^2 m^3}{12(8\pi)^3 M_P^2}, \tag{90}$$

where y_t denotes the top Yukawa coupling, and $n_{t,H}$ are integer modular weights. The decay to gravitinos is in this case

$$\Gamma(T \to \psi_{3/2}\psi_{3/2}) \sim 10^{-3} \left(\frac{m_{3/2}}{m}\right)^2 \frac{m^3}{M_P^2} \tag{91}$$

up to factors at most $\mathcal{O}(1)$ that are dependent on the details of the supersymmetry-breaking sector. In this case the branching ratio into gravitinos is negligibly small.

6 Implications for Supersymmetric Inflationary Models

We now consider the implications of our results for supersymmetric models of inflation. Our best estimate of post-inflationary gravitino production corresponds to the estimate (49). We assume that the gravitino is not the LSP, but that it is heavy enough to decay into MSSM particles, and confront (49) with constraints from Big-Bang nucleosynthesis (BBN) [26, 27, 16, 22] and the relic cold dark matter density.

Standard BBN calculations are in good agreement with the measured light-element abundances, with the apparent exception of Lithium [53]. One may use this agreement to set an upper bound on the gravitino abundance, or one may postulate that the gravitino abundance saturates the upper bound, in which case gravitino decays may mitigate the cosmological Lithium problem. Studies in a variety of supersymmetric models compatible with LHC and other constraints yielded [27]

$$\zeta_{3/2} \equiv m_{3/2} \frac{n_{3/2}}{n_\gamma} = \frac{m_{3/2}}{2} Y_{3/2} \lesssim 10^{-11} \, \text{GeV} \quad \text{for} \quad m_{3/2} \sim 3 \, \text{TeV}, \tag{92}$$

rising to $\zeta_{3/2} \lesssim 10^{-8}$ GeV for $m_{3/2} \sim 6$ TeV. Combining (48) and (92), we find

$$\Gamma_\phi \lesssim \left(1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2}\right)^{-2} \begin{cases} 8.3 \times 10^{-6} \, \text{GeV} & \text{for} \quad m_{3/2} = 3 \, \text{TeV}, \\ 2.1 \, \text{GeV} & \text{for} \quad m_{3/2} = 6 \, \text{TeV}. \end{cases} \tag{93}$$

In the case of two-body inflaton decay via a superpotential coupling y, one has $\Gamma_\phi = |y|^2 m/8\pi$. Assuming $m \simeq 10^{-5} M_P$, the bounds (93) correspond to

$$|y| \lesssim \left(1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2}\right)^{-1} \begin{cases} 2.9 \times 10^{-9} & \text{for} \quad m_{3/2} = 3 \, \text{TeV}, \\ 1.5 \times 10^{-6} & \text{for} \quad m_{3/2} = 6 \, \text{TeV}. \end{cases} \tag{94}$$

These bounds would be weakened for larger values of $m_{3/2}$, disappearing altogether if the gravitino is sufficiently heavy to decay before BBN.

However, there is another bound on $Y_{3/2}$ that applies even in this case, which comes from the contribution to the cold dark matter density from supersymmetric relic dark matter particles with mass $m_{\rm LSP}$ produced as the end-products of inflaton decay [54]:

$$Y_{3/2} \;\; < \;\; \frac{2\,\Omega_{\rm cold}\,\rho_c}{m_{\rm LSP}\,n_\gamma}\,, \tag{95}$$

where ρ_c is the closure density and the factor of 2 is present because we have defined $Y_{3/2}$ in terms of $n_{\rm rad} = n_\gamma/2$. Using $\Omega_{\rm cold}h^2 = 0.120$ and $\rho_c = 1.054 \times 10^{-5}h^2\,{\rm cm}^{-3}{\rm GeV}$, one finds

$$Y_{3/2} \;\; < \;\; 6.16 \times 10^{-9} \left(\frac{\rm GeV}{m_{\rm LSP}}\right)\,. \tag{96}$$

Using (48), this bound corresponds to

$$|y| \;\; < \;\; 2.7 \times 10^{-5} \left(1 + 0.56\,\frac{m_{1/2}^2}{m_{3/2}^2}\right)^{-1} \left(\frac{100{\rm GeV}}{m_{\rm LSP}}\right)\,, \tag{97}$$

and we see that $|y| \lesssim 10^{-5}$ for plausible LSP masses in the range of a few hundred GeV, as assumed in [6].

It was calculated in [6] that $|y| \lesssim 10^{-5}$ corresponds, for Starobinsky-like inflationary models, to a number of inflationary e-folds $N_* \lesssim 52$. This can to be compared with the 68% lower limit $N_* \gtrsim 50$ from the Planck 2015 constraint on the tilt of the scalar perturbation spectrum. According to the calculations in [6], the stronger BBN bound in (94) for $m_{3/2} = 3$ TeV would correspond to $N_* \lesssim 49$, outside the Planck 2015 68% CL range for Starobinsky-like models of inflation, though within the 95% CL range $N_* \gtrsim 44$. Supersymmetric Starobinsky-like models are clearly coming under pressure.

We conclude this Section with a discussion of non-thermally-produced gravitinos. The bound (97) assumes that all gravitinos are produced by scatterings in the relativistic plasma produced during the decay of the inflaton. In Section 5 we also computed the amount of gravitinos produced by the hard inflaton decay products (before they thermalize), and those produced directly in inflaton decays. While the former effect is negligible, the latter is model-dependent and gives an additional gravitino population with abundance given by (83). In this case the bound (96) from the dark matter abundance on $Y_{3/2,{\rm from\,decay}}$, given by (82), translates into

$$B_{3/2}|y| < 5.2 \times 10^{-11} \left(\frac{m}{M_P}\right)^{1/2} \left(\frac{100{\rm GeV}}{m_{\rm LSP}}\right) \simeq 1.7 \times 10^{-13} \left(\frac{100{\rm GeV}}{m_{\rm LSP}}\right)\,. \tag{98}$$

In this relations $B_{3/2}$ is the branching ratio of the inflaton decays to gravitinos, which is model-dependent. For example, as discussed in Subsection (5.2), in the case of an untwisted sneutrino inflaton with Starobinsky potential studied in [39], a significant branching ratio can be induced by superpotential terms that couple the inflaton ϕ to the volume modulus

T responsible for supersymmetry breaking, of the form $W \supset \zeta(T - 1/2)^2\phi$. In this case, assuming that the dominant channels correspond to the matter-Higgs decays (84), the bound (98) translates into the stringent upper limit

$$|\zeta| < 8.7 \times 10^{-7} |y|^{1/2} \left(\frac{100\text{GeV}}{m_\text{LSP}}\right)^{1/2} < 4.5 \times 10^{-9} \left(1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2}\right)^{-1/2} \left(\frac{100\text{GeV}}{m_\text{LSP}}\right). \quad (99)$$

7 Summary

In this paper we have revisited gravitino production in supersymmetric models of inflation, with applications to Starobinsky-like models. Our main focus has been to examine the production of gravitinos during inflaton decay, before reheating is completed. Under the assumption of instantaneous thermalization (but not instantaneous inflaton decay), the dilute thermal bath formed by the inflaton decay products reaches a maximum temperature while the bulk of the inflaton density has yet to decay. This produces a temporary large abundance of gravitinos. However, these gravitinos are diluted away as the bulk of the inflaton density subsequently decays, so that naive estimates of the gravitino production based on instantaneous inflaton decay turn out to be rather accurate [19].

In this work we extended previous studies by providing some semi-analytic and analytic results for the final abundance. We started in Section 2 by obtaining a simple parametrization of the rate of inverse decay processes that contribute to the gravitino abundance [25]. While the full analytic expression of these rates is rather involved, we show that they can be parametrized accurately, as done for the thermal scatterings in [23], for instance. Therefore, we can use the same parametrization as [23], though with different numerical values for the parameters c_i and k_i so as to include the additional processes pointed out in [25]. This leads us to the solution (19) under the assumption of instantaneous inflaton decay. In this expression, we include an arbitrary parameter c to describe the time of this supposedly instantaneous decay:

$$t \equiv c/\Gamma_\phi, \quad (100)$$

where Γ_ϕ is the total inflaton decay rate. The value of c cannot be obtained from this simplified computation, and the only way to obtain accurately the gravitino abundance is to include the precise evolution of the inflaton and of the thermal number densities during the inflaton decay [25].

We performed this study in Section 3. Our parametrization of the gravitino production rate allows one to obtain an accurate analytic solution for the gravitino abundance, Eq. (43). To solve this equation analytically, we use the numerical solution obtained in [6] through an iteration procedure, which was shown to be very accurate. We thus obtain the analytic solution (46). To verify the accuracy of the solution, we also solved the full system of equations numerically, obtaining the result (49). From this result, we could see that the analytic solution (46) is accurate at the $\lesssim 2\%$ level over the entire range $\Gamma_\phi \lesssim 10^{-7} M_p$.

The instantaneous decay approximation ignores (i) the gravitino quanta produced before c/Γ_ϕ, and (ii) the dilution of these quanta by subsequent inflation decays. If too small a value

of c is chosen, an overestimate of the correct abundance results from (ii). On the other hand, if too large a value of c is chosen, an underestimate of the correct abundance results from (i). It follows that there must be some value of c that provides the correct result *accidentally*. The comparison between our results (19) and (49) shows that this accidental value is $c \simeq 1.2$ (in agreement with [24]). In Fig. 5 we compare the evolution of the gravitino abundance in the exact case vs. the instantaneous inflaton decay case, showing how the latter provides an overestimate of the correct result if $c = 1$ is chosen.

To obtain a complete answer on the gravitino abundance, we extended this study in two directions. In Section 4 we checked whether relaxing the assumption of instantaneous thermalization of the inflaton decay products modifies the final gravitino abundance. For Planck-suppressed inflaton decays, thermalization is mainly due to small-angle scatterings that increase the number of particles [41]. We verified that the maximum temperature of the thermal bath is in general lower than the result (36), since thermalization is generically delayed to later times, $v_{\rm th} > v_{\rm max}$. However, this does not affect the final gravitino abundance, as thermalization is in any case achieved well before $\Gamma_\phi t = \mathcal{O}(1)$, which is when most of the gravitino abundance is generated.

As a second extension, in Section 5 we computed the amount of non-thermal production of gravitinos. We first studied the production from hard quanta generated by the inflaton decay, before thermalization takes place. The motivation for this analysis is that the production rate for gravitinos from scatterings increases with the energy of the incoming particles, and the energy of such quanta is a $\mathcal{O}(1)$ fraction of the inflaton mass. However, we find that the gravitinos produced by these quanta are diluted by the subsequent inflaton evolution and decays, and provide a negligible contribution to the final gravitino abundance. We also considered the gravitinos directly produced by inflaton decays. This is clearly a model-dependent study and, after a general discussion, we focused on no-sale supergravity models that can accommodate Starobinsky-like inflation. We find that the direct gravitino production is generally subdominant with respect to the thermal one, with the exception of contributions due to a specific superpotential coupling (87) of the inflaton to a supersymmetry-breaking modulus field T. This does not modify the inflaton potential, and is not constrained by inflation.

Finally, in Section 6 we studied the phenomenological implications of gravitino production. As is well known, gravitino production following inflation is subject to two important constraints: late-decaying gravitinos may destroy the agreement of standard BBN calculations with astrophysical measurements of light-element abundances, and supersymmetric dark matter particles (LSPs) produced in gravitino decays may have a density exceeding cosmological and astrophysical limits. We have expressed these constraints in terms of a generic two-body inflaton decay coupling y. The BBN constraint enforces the bound $y \lesssim 2.9 \times 10^{-9}$ for $m_{3/2} = 3$ TeV, which is relaxed for larger gravitino masses, see (94). As seen in (97), the dark matter density constraint enforces $y \lesssim 10^{-5}$ for LSP masses of a few hundred GeV. Within the context of Starobinsky-like models of inflation [6], such as many models based on no-scale supergravity, the former, stronger constraint would correspond to a number of inflationary e-folds $N_* \lesssim 49$, outside the Planck 68% CL range for such models, $N_* \gtrsim 50$, whereas the latter, weaker limit would correspond to $N_* \lesssim 52$, still compatible with the

Planck 68% CL range.

The analysis of this paper strengthens the potential connections between inflationary cosmology and TeV-scale physics. The decays of the inflaton into 'light' particles are constrained by gravitino production as well as CMB measurements, and gravitino production is in turn constrained by BBN and the density of cold dark matter, by an amount that depends on the mass of the lightest supersymmetric particle, as seen in (97). The constraints are becoming quite tight, implying that either some observable signal should soon turn up, or the supersymmetric inflationary framework discussed here may need to be rethought.

Acknowledgements

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