

# Massive Higher Spin Fields Coupled to a Scalar: Aspects of Interaction and Causality

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## Abstract

We consider in detail the most general cubic Lagrangian which describes an interaction between two identical higher spin fields in a triplet formulation with a scalar field, all fields having the same values of the mass. After performing the gauge fixing procedure we find that for the case of massive fields the gauge invariance does not guarantee the preservation of the correct number of propagating physical degrees of freedom. In order to get the correct number of degrees of freedom for the massive higher spin field one should impose some additional conditions on parameters of the vertex. Further independent constraints are provided by the causality analysis, indicating that the requirement of causality should be imposed in addition to the requirement of gauge invariance in order to have a consistent propagation of massive higher spin fields.

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# 1 Introduction

It is already well known that massless higher spin fields can interact consistently on an arbitrary dimensional anti-de Sitter  $AdS_{\mathcal{D}}$  background [1]. On the other hand, the situation is quite different as far as interacting theories for higher spin fields on a flat background are concerned. The corresponding cubic interaction vertices for massless fields, whose study has been initiated in [2]–[3], are fairly well understood now [4]–[10]. The application of cubic vertices on a flat background can be twofold. The first one is to use them for a further deformation to  $AdS_{\mathcal{D}}$ , thus studying the cubic interactions in the “metric like” formalism [11]–[13].<sup>§</sup>

Another important point is to understand if one can have a consistent interacting theory of massless and massive point particles beyond the cubic level on a flat background. To this end one can proceed by constructing quartic interaction vertices and study their properties [6]–[7], [17]–[19]. The study of quartic vertices for massless higher spin fields revealed that although the corresponding quartic Lagrangian is gauge invariant, the symmetry of the scattering amplitudes requires the inclusion of extra composite/nonlocal objects into the theory, otherwise the corresponding S-matrix is trivial even if the number of fields under consideration is infinite [7] (let us note also an interesting application of the composite objects in the framework of the  $AdS_4/CFT_3$  correspondence [20]). One can therefore draw the conclusion that gauge invariance itself is not a sufficient requirement for the consistency of the interacting theory: rather one should perform some extra tests to investigate if the interacting theory is consistent or not.

Apart from the requirement of having nontrivial S-matrix for the theory of higher spin fields on a flat background, one can employ the Velo-Zwanziger causality consistency test if the fields under consideration have non-zero mass. According to [21] – [22] massive fields already with spin one interacting with some nontrivial background can exhibit noncausal propagation, hence violating consistency of the theory. Obviously the same kind of difficulty can appear for massive higher spin fields as well and checks for different systems have been performed [23]–[29]. In particular in [23] it has been shown that the massive fields with spin 2 propagating on an anti-de Sitter background do not violate causality. Another famous example of interacting massive higher spin fields which preserve causality is String Theory. The Velo-Zwanziger problem in the framework of String Theory has been considered in detail in [28], where it was shown how this problem is avoided.

It is interesting to point out that String Theory is not the only example of a theory of interacting higher spin fields with non-zero mass which exhibits causal propagation. The results of [29] show that massive higher spin fields interacting with a background constant electromagnetic field can also avoid Velo-Zwanziger inconsistency at least at linear order in electromagnetic field, thus revealing a causal propagation. Causality for massive spin 3/2 coupled to an electromagnetic field to

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<sup>§</sup>Cubic and higher-order interactions in the “frame-like” formalism have been extensively discussed in [14]–[16].

all orders in a constant field strength has been proven in [26], and causality for spin 2 and spin 3/2 fields on a gravitational background has been discussed in [27] in the framework of the BRST approach. Finally in the recent paper [30] it has been argued that even for a massless theory the gauge invariance itself might not be a sufficient argument for consistency at the interacting level, therefore the causality again should be checked separately.

In the present paper we consider the cubic interaction of massive higher spin fields with a scalar field. For simplicity we take the masses of all fields to be the same. The consideration is actually performed in two steps. The first step is to check that the inclusion of the nonlinear cubic interactions into a system which describes free massive scalar and free massive higher spin fields does not change the number of original degrees of freedom. As we shall see, already this requirement can impose some strong constraints on the free parameters of interaction and on the mass parameter. Provided that this requirement is satisfied, the second step is to perform the causality test for this system. A completely rigorous analysis of the causality in the model under consideration is very complicated. Moreover, it is not very clear how it can be done in the case when the number of derivatives in vertices is greater than the number of derivatives in a free action. In this paper, we propose a simplified model allowing us to apply the Velo-Zwanziger procedure for causality analysis. In particular since this procedure assumes that the number of derivatives in the action is not higher than two, we shall consider some kind of low-energy approximation and keep only those terms in the vertex which contain at most the second derivative acting on the higher spin field. Besides, the scalar field will be considered as external background. As a result we get a dynamical higher spin field coupled to an external scalar field and all derivatives acting on the dynamical field are at most of the second order. Then we shall see that causality analysis imposes some additional requirements on the interaction structure.

We find that it is quite difficult to satisfy both (the preservation of the correct degrees of freedom and causality) requirements. In particular the allowed solutions of the first test have been excluded by the second one and vice versa. The only allowed option is when the mass parameter  $m$  is sufficiently large, which allows us to ignore certain terms in the original action without imposing extra conditions on the background. After that we find from the causality analysis for our simplified model that for certain choices of coupling constants, or more precisely for certain choices of parameters entering the cubic interaction vertex, the causality is preserved, whereas for the other choices of these parameters the causality is broken.

To summarize, we proceed as follows. First we consider a relevant cubic Lagrangian which contains two identical higher spin fields and one scalar. We consider the scalar to be a background field, whereas the higher spin field is taken to be dynamical. Then we perform a gauge fixing procedure to obtain an on-shell cubic interaction vertex and take a low-energy approximation, i.e. keep only the terms in the Lagrangian which contain a maximum of two derivatives acting on the dynamical higher spin fields. After this, finally we perform the Velo-Zwanziger like analysis

for the system, i.e. compute the characteristic determinant  $D(p)$  and find for which values of the free parameters this determinant contains the second derivatives acting on the dynamical field only in the form of the d’Alembertian.

Let us stress that the systems which contain dynamical fields with spin greater than two turn out to be the ones where the procedure described above is nontrivial in the following sense. For the massive fields with spin one and spin two the causality and the correct number of physical degrees of freedom can be preserved for a constant background scalar field, which in turn means a simple redefinition of the mass parameter in the theory. For this reason, in the paper we start from the first nontrivial example, i.e. from the  $3 - 3 - 0$  system, and leave a more detailed discussion of lower spin fields for the Appendices A–B. In the case of dynamical higher spin fields the background scalar is no longer constant, and since it couples to the dynamical higher spin fields via the derivatives, one has effectively a Lorentz-violating background. This is similar to other examples of the Velo-Zwanziger problem considered previously in the literature, although in our case a nontrivial background scalar field is involved rather than, say, a nontrivial electromagnetic background field.

In this paper we will be using the reducible symmetrical representations of the Poincaré group, since an off-shell formulation for them is simpler than an off-shell formulation for irreducible higher spin modes (so-called “triplet” [31]). Therefore below whenever we say “a massless triplet with spin  $s$ ” we actually mean a symmetric tensor field of rank  $s$  along with auxiliary fields with ranks  $s - 1$  and  $s - 2$ . The physical polarizations of a triplet contains fields with spins  $s, s - 2, \dots, 1/0$  with their masses equal to zero. Similarly a massive triplet (massive reducible representation of the Poincaré group) is described by a symmetric field of rank  $s$  along with some auxiliary fields. The physical polarizations are again fields with spins  $s, s - 2, \dots, 1/0$  with the same value of mass<sup>¶</sup>. It would be very interesting to generalize our present analysis for for some other systems such as fields with half integer spin or fields with mixed symmetries interacting with some nontrivial background [33] (see for example [34] for recent progress for mixed symmetry fields). We hope to come back to this issue in future.

The paper is organized as follows.

As a preparation for the massive case, in Section 2 we give an explicit example of a massless field with spin three interacting with a massless scalar in the triplet formulation. Since we are using an off-shell formulation, the Lagrangian and equations of motion will contain both physical and auxiliary fields. These auxiliary fields, which we denote as  $|C\rangle$  and  $|D\rangle$ , are the feature of the Lagrangian BRST formulations of the higher spin fields [3], [7], [9]– [11] and are absent in the on-shell vertices. We present in detail the derivation of the Lagrangian, of the equations of motion, and of the gauge transformations for this system, and show that the number of physical degrees of freedom is preserved after nonlinear deformation of the free equations.

In Section 3 we carry out an analogous procedure for the system which contains

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<sup>¶</sup>Triplet formulation of higher spin fields can be further generalized to get completely unconstrained “quartet” formulation [32].

two identical massive fields with spin three and a massive scalar. After carrying out the gauge fixing procedure for the cubic Lagrangian, we find that the transversality condition can be violated, unless one imposes extra conditions on the parameters of the theory and on the background. It means that the gauge invariance in massive case itself does not guarantee preservation of the correct number of degrees of freedom.

In Section 4 we consider the aspects of causality. We formulate a simplified model allowing us to perform the Velo-Zwanziger like analysis for the  $3 - 3 - 0$  system and in Section 5 we generalize these results for the case of the  $s - s - 0$  system.

The last section contains our conclusions.

Finally, Appendix A contains detailed expressions for the first-order gauge transformations in Section 3 and Appendix B contains a discussion of the  $1 - 1 - 0$  and  $2 - 2 - 0$  systems.

## 2 3-3-0 Vertex: Massless Fields

The goal of this section is to consider some details of a cubic interaction between two massless spin 3 triplets with with a massless scalar. We demonstrate that nonlinear corrections to the free equations of motion and to the gauge transformations do not change the number of physical polarizations. In other words we check that in the massless case the requirement of gauge invariance is sufficient to construct the cubic vertex which preserves the correct number of degrees of freedom.

### 2.1 Fields and parameters

An off-shell cubic vertex for two triplets with spin 3 and one scalar can be obtained from the vertex (see [3], [7] for details)

$$|V\rangle = e^{Y_\alpha^+} e^{Y_{gh}^+} c_0^{(1)} c_0^{(2)} c_0^{(3)} |0\rangle_{123}, \quad (2.1)$$

where

$$Y_\alpha^+ = a_1(\alpha^{(1)+} \cdot (p^{(2)} - p^{(3)}) + \alpha^{(2)+} \cdot (p^{(3)} - p^{(1)}) + \alpha^{(3)+} \cdot (p^{(1)} - p^{(2)})), \quad (2.2)$$

$$Y_{gh}^+ = a_1(c^{(1)+}(b_0^{(3)} - b_0^{(2)}) + c^{(2)+}(b_0^{(1)} - b_0^{(3)}) + c^{(3)+}(b_0^{(2)} - b_0^{(1)})). \quad (2.3)$$

Here  $a_1$  is an arbitrary constant and  $A \cdot B \equiv A_\mu B^\mu$ . The operator  $p_\mu^{(i)}$  is a derivative acting on the fields in the  $i^{\text{th}}$  Hilbert space. They have the form  $p_\mu^{(i)} = -i\partial_\mu$  when acting on the right and  $p_\mu^{(i)} = i\partial_\mu$  when acting on the left. The oscillators obey standard (anti-)commutation relations

$$[\alpha_\mu^{(i)}, \alpha_\nu^{(j)+}] = \eta_{\mu\nu} \delta^{ij}, \quad \{c_0^{(i)}, b_0^{(j)}\} = \{c^{(i)+}, b^{(j)}\} = \{c^{(i)}, b^{(j)+}\} = \delta^{ij}. \quad (2.4)$$

Since we are considering an interaction of the type  $s - s - 0$  the corresponding cubic vertex contains a maximal number of derivatives. The vertex (2.2)–(2.3) has

a cyclic symmetry, and so we take the higher spin functional and the parameter of gauge transformations to be of the form.<sup>||</sup>

$$|\Phi_i\rangle = \frac{1}{3!}\phi_{\mu_1\mu_2\mu_3}(x)\alpha_{\mu_1}^{(i)+}\alpha_{\mu_2}^{(i)+}\alpha_{\mu_3}^{(i)+}|0\rangle_i - \frac{i}{2!}C_{\mu_1\mu_2}(x)\alpha_{\mu_1}^{(i)+}\alpha_{\mu_2}^{(i)+}c_0^{(i)}b^{(i)+}|0\rangle_i + D_{\mu_1}(x)\alpha_{\mu_1}^{(i)+}c^{(i)+}b^{(i)+}|0\rangle_i + \phi(x)|0\rangle_i, \quad (2.5)$$

$$|\Lambda_i\rangle = \frac{i}{2!}\lambda_{\mu_1\mu_2}(x)\alpha_{\mu_1}^{(i)+}\alpha_{\mu_2}^{(i)+}b^{(i)+}|0\rangle_i, \quad (2.6)$$

or in a more compact form as

$$|\Phi_i\rangle = |\phi_i\rangle + c^{(i)+}b^{(i)+}|D_i\rangle + c_0^{(i)}b^{(i)+}|C_i\rangle, \quad |\Lambda_i\rangle = b^{(i)+}|\lambda_i\rangle. \quad (2.7)$$

The nilpotent BRST charges for each Hilbert space are

$$Q^{(i)} = c_0^{(i)}l_0^{(i)} + c^{(i)}l^{(i)+} + c^{(i)+}l^{(i)} - c^{(i)+}c^{(i)}b_0^{(i)}, \quad i = 1, 2, 3, \quad (2.8)$$

where we used the notation

$$l_0^{(i)} = p^{(i)} \cdot p^{(i)}, \quad l^{(i)+} = p^{(i)} \cdot \alpha^{(i)+}, \quad l^{(i)} = p^{(i)} \cdot \alpha^{(i)}. \quad (2.9)$$

Finally the cubic Lagrangian has the form

$$L = \sum_{i=1}^3 \int dc_0^{(i)} \langle \Phi_i | Q^{(i)} | \Phi_i \rangle + g \left( \int dc_0^{(1)} dc_0^{(2)} dc_0^{(3)} \langle \Phi_1 | \langle \Phi_2 | \langle \Phi_3 | | V_3 \rangle + h.c. \right). \quad (2.10)$$

The Lagrangian (2.10) is invariant under the gauge transformations

$$\delta|\Phi_i\rangle = Q^{(i)}|\Lambda_i\rangle - g \int dc_0^{(i+1)} dc_0^{(i+2)} [(\langle \Phi_{i+1} | \langle \Lambda_{i+2} | + \langle \Phi_{i+2} | \langle \Lambda_{i+1} |) | V_3 \rangle], \quad (2.11)$$

up to linear order in the coupling constant  $g$  due to the nilpotency of each BRST charge  $(Q^{(i)})^2 = 0$  and the BRST invariance of vertex (2.1)

$$\sum_i^3 Q^{(i)}|V\rangle = 0. \quad (2.12)$$

## 2.2 Gauge transformations

In the notation of (2.7) we have for gauge transformations:

$$\begin{aligned} & \delta|\phi_1\rangle + c^{(1)+}b^{(1)+}\delta|d_1\rangle + c_0^{(1)}b^{(1)+}\delta|C_1\rangle \\ &= l^{(1)+}|\lambda_1\rangle + c_0^{(1)}b^{(1)+}l_0^{(1)}|\lambda_1\rangle + c^{(1)+}b^{(1)+}l^{(1)}|\lambda_1\rangle \\ &+ [-ga_1\langle\phi_2|\langle\lambda_3| + ga_1\langle\phi_3|\langle\lambda_2| + ga_1^2\langle C_3|\langle\lambda_2| + ga_1^2\langle C_2|\langle\lambda_3|]e^{Y_{\alpha^+}}|0\rangle_{123}. \end{aligned} \quad (2.13)$$

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<sup>||</sup>In the rest of this subsection no summation over the repeated indices is assumed, unless explicitly specified.

Therefore

$$\delta C_{\mu\nu} = \square \lambda_{\mu\nu}, \quad (2.14)$$

$$\delta D_\mu = \partial^\nu \lambda_{\mu\nu}, \quad (2.15)$$

where  $\square \equiv \partial^\mu \partial_\mu$ .

The gauge transformations for  $\phi_{\mu_1\mu_2\mu_3}$  and  $\phi$  are more complicated

$$\begin{aligned} & \left( \frac{1}{3!} \delta \phi_{\mu_1\mu_2\mu_3} \alpha_{\mu_1}^{(1)+} \alpha_{\mu_2}^{(1)+} \alpha_{\mu_3}^{(1)+} + \delta \phi \right) |0\rangle_1 = \\ & \frac{1}{3!} \partial_{(\mu_1} \lambda_{\mu_2\mu_3)} \alpha_{\mu_1}^{(1)+} \alpha_{\mu_2}^{(1)+} \alpha_{\mu_3}^{(1)+} |0\rangle_1 + g a_1 [-\langle \phi_2 | \langle \lambda_3 | + \langle \phi_3 | \langle \lambda_2 |] e^{Y_\alpha^+} |0\rangle_{123}. \end{aligned} \quad (2.16)$$

The term proportional to  $g$  in (2.16) is a nonabelian deformation of the gauge transformations for  $\phi_{\mu_1\mu_2\mu_3}$

$$\begin{aligned} & -\langle \phi_2 | \langle \lambda_3 | e^{Y_\alpha^+} |0\rangle_{123} + \langle \phi_3 | \langle \lambda_2 | e^{Y_\alpha^+} |0\rangle_{123} = \\ & + {}_2\langle 0 | {}_3\langle 0 | \phi \frac{i}{2} (\lambda_{\nu_1\nu_2} \alpha_{\nu_1}^{(3)} \alpha_{\nu_2}^{(3)}) \frac{1}{2} (a_1 \alpha^{(3)+} \cdot (p^{(1)} - p^{(2)}))^2 \\ & \quad \times \frac{1}{3!} (a_1 \alpha^{(1)+} \cdot (p^{(2)} - p^{(3)}))^3 |0\rangle_{123} \\ & - {}_3\langle 0 | {}_2\langle 0 | \phi \frac{i}{2} (\lambda_{\nu_1\nu_2} \alpha_{\nu_1}^{(2)} \alpha_{\nu_2}^{(2)}) \frac{1}{2} (a_1 \alpha^{(2)+} \cdot (p^{(3)} - p^{(1)}))^2 \\ & \quad \times \frac{1}{3!} (a_1 \alpha^{(1)+} \cdot (p^{(2)} - p^{(3)}))^3 |0\rangle_{123}. \end{aligned} \quad (2.17)$$

The operator  $p_\mu^{(1)}$  in the equation (2.17) should be replaced with  $-p_\mu^{(2)} - p_\mu^{(3)}$  due to the relation

$$p_\mu^{(1)} + p_\mu^{(2)} + p_\mu^{(3)} = 0, \quad (2.18)$$

which reflects the fact that one can discard the total derivative in the Lagrangian. This is justified in the equations of motion and gauge transformation rules since they are Lagrangian equations and represent invariance of a Lagrangian.

Finally one obtains for the tensor field

$$\begin{aligned} & \frac{1}{3!} \delta \phi_{\mu_1\mu_2\mu_3} = \frac{1}{3!} \partial_{(\mu_1} \lambda_{\mu_2\mu_3)} - \\ & \frac{g a_1^6}{2} [4(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2} \phi) \lambda_{\nu_1\nu_2} + 4(\partial_{\mu_1\mu_2\mu_3\nu_1} \phi) (\partial_{\nu_2} \lambda_{\nu_1\nu_2}) + (\partial_{\mu_1\mu_2\mu_3} \phi) (\partial_{\nu_1\nu_2} \lambda_{\nu_1\nu_2}) \\ & - 12(\partial_{\mu_1\mu_2\nu_1\nu_2} \phi) (\partial_{\mu_3} \lambda_{\nu_1\nu_2}) - 12(\partial_{\mu_1\mu_2\nu_1} \phi) (\partial_{\nu_2\mu_3} \lambda_{\nu_1\nu_2}) - 3(\partial_{\mu_1\mu_2} \phi) (\partial_{\nu_1\nu_2\mu_3} \lambda_{\nu_1\nu_2}) \\ & + 12(\partial_{\mu_1\nu_1\nu_2} \phi) (\partial_{\mu_2\mu_3} \lambda_{\nu_1\nu_2}) + 12(\partial_{\mu_1\nu_1} \phi) (\partial_{\mu_2\mu_3\nu_2} \lambda_{\nu_1\nu_2}) + 3(\partial_{\mu_1} \phi) (\partial_{\mu_2\mu_3\nu_1\nu_2} \lambda_{\nu_1\nu_2}) \\ & - 4(\partial_{\nu_1\nu_2} \phi) (\partial_{\mu_1\mu_2\mu_3} \lambda_{\nu_1\nu_2}) - 4(\partial_{\nu_1} \phi) (\partial_{\mu_1\mu_2\mu_3\nu_2} \lambda_{\nu_1\nu_2}) - \phi (\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2} \lambda_{\nu_1\nu_2})], \end{aligned} \quad (2.19)$$

where the nonlinear terms on the right hand-side of the equation (2.19), as well as in all analogous equations below, are assumed to be symmetrized with weight 1 with

respect to the free indices. Similarly for the scalar one has

$$\begin{aligned}
\delta\phi = & -\frac{ga_1^6}{3!} [32(\partial_{\rho_1\rho_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2\nu_3}\lambda_{\rho_1\rho_2}) + 32(\partial_{\rho_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2\nu_3\rho_2}\lambda_{\rho_1\rho_2}) \\
& + 8\phi_{\nu_1\nu_2\nu_3}(\partial_{\nu_1\nu_2\nu_3\rho_1\rho_2}\lambda_{\rho_1\rho_2}) + 48(\partial_{\nu_1\rho_1\rho_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3}\lambda_{\rho_1\rho_2}) \\
& + 48(\partial_{\nu_1\rho_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3\rho_2}\lambda_{\rho_1\rho_2}) + 12(\partial_{\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3\rho_1\rho_2}\lambda_{\rho_1\rho_2}) \\
& + 24(\partial_{\nu_1\nu_2\rho_1\rho_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3}\lambda_{\rho_1\rho_2}) + 24(\partial_{\nu_1\nu_2\rho_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3\rho_2}\lambda_{\rho_1\rho_2}) \\
& + 6(\partial_{\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3\rho_1\rho_2}\lambda_{\rho_1\rho_2}) + 4(\partial_{\nu_1\nu_2\nu_3\rho_1\rho_2}\phi_{\nu_1\nu_2\nu_3})\lambda_{\rho_1\rho_2} \\
& + 4(\partial_{\nu_1\nu_2\nu_3\rho_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\rho_2}\lambda_{\rho_1\rho_2}) + (\partial_{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\rho_1\rho_2}\lambda_{\rho_1\rho_2})] \\
& + \frac{ga_1^6}{2} [16(\partial_{\mu_1\mu_2}\lambda_{\nu_1\nu_2})(\partial_{\nu_1\nu_2}C_{\mu_1\mu_2}) + 16(\partial_{\mu_1\mu_2\nu_1}\lambda_{\nu_1\nu_2})(\partial_{\nu_2}C_{\mu_1\mu_2}) \\
& + 4(\partial_{\mu_1\mu_2\nu_1\nu_2}\lambda_{\nu_1\nu_2})C_{\mu_1\mu_2} + 16(\partial_{\mu_1}\lambda_{\nu_1\nu_2})(\partial_{\mu_2\nu_1\nu_2}C_{\mu_1\mu_2}) \\
& + 16(\partial_{\mu_1\nu_1}\lambda_{\nu_1\nu_2})(\partial_{\mu_2\nu_2}C_{\mu_1\mu_2}) + 4(\partial_{\mu_1\nu_1\nu_2}\lambda_{\nu_1\nu_2})(\partial_{\mu_2}C_{\mu_1\mu_2}) \\
& + 4\lambda_{\nu_1\nu_2}(\partial_{\mu_1\mu_2\nu_1\nu_2}C_{\mu_1\mu_2}) + 4(\partial_{\nu_2}\lambda_{\nu_1\nu_2})(\partial_{\mu_1\mu_2\nu_1}C_{\mu_1\mu_2}) + (\partial_{\nu_2\nu_2}\lambda_{\nu_1\nu_2})(\partial_{\mu_1\mu_2}C_{\mu_1\mu_2})].
\end{aligned} \tag{2.20}$$

### 2.3 Equations of Motion

From the Lagrangian (2.10) written in terms of component fields (2.7), i.e. after integrating out the ghost variables

$$\begin{aligned}
L = & \sum_{i=1,2,3} \left( \langle \phi_i | l_0^{(i)} | \phi_i \rangle - \langle D_i | l_0^{(i)} | D_i \rangle + \langle C_i | | C_i \rangle - \langle \phi^i | l^{(i)+} | C^i \rangle + \langle D_i | l^{(i)} | C_i \rangle \right. \\
& \left. - \langle C_i | l^{(i)} | \phi_i \rangle + \langle C_i | l^{(i)+} | D_i \rangle \right) \\
& - g \left[ \langle \phi_3 | \langle \phi_2 | \langle \phi_1 | + a_1^2 \langle C_3 | \langle C_2 | \langle \phi_1 | + a_1^2 \langle C_2 | \langle C_1 | \langle \phi_3 | + a_1^2 \langle C_3 | \langle C_1 | \langle \phi_2 | \right) e^{Y_\alpha^+} | 0 \rangle_{123} \\
& + h.c. ],
\end{aligned} \tag{2.21}$$

one can readily derive the corresponding equations of motion.

The equation of motion with respect to  $\langle \phi_1 |$  :

$$l_0^{(1)} | \phi_1 \rangle - l^{(1)+} | C_1 \rangle - g \langle \phi_3 | \langle \phi_2 | e^{Y_\alpha^+} | 0 \rangle_{123} - ga_1^2 \langle C_3 | \langle C_2 | e^{Y_\alpha^+} | 0 \rangle_{123} = 0. \tag{2.22}$$

Let us note that this expression actually contains two equations: one is with respect to  $\phi_{\mu_1\mu_2\mu_3}$  and the other is with respect to  $\phi$ .

The equation of motion with respect to  $\langle C_1 |$ :

$$| C_1 \rangle - l^{(1)} | \phi_1 \rangle + l^{(1)+} | D_1 \rangle - ga_1^2 (\langle C_2 | \langle \phi_3 | + \langle C_3 | \langle \phi_2 |) e^{Y_\alpha^+} | 0 \rangle_{123} = 0, \tag{2.23}$$

and finally the equation of motion with respect to  $\langle D_1 |$ :

$$l_0^{(1)} | D_1 \rangle - l^{(1)} | C_1 \rangle = 0. \tag{2.24}$$

Let us first consider the equation with respect to  $\phi_{\mu_1\mu_2\mu_3}$ . It contains two parts:

$$l_0^{(1)}|\phi_1\rangle - l^{(1)+}|C_1\rangle = [-\square\phi_{\nu_1\nu_2\nu_3} + \partial_{(\nu_1}C_{\nu_2\nu_3)}] \frac{1}{3!}\alpha_{\nu_1}^{(1)+}\alpha_{\nu_2}^{(1)+}\alpha_{\nu_3}^{(1)+}|0\rangle_1, \quad (2.25)$$

and

$$\begin{aligned} & -g\langle\phi_3|\langle\phi_2|e^{Y_\alpha^+}|0\rangle_{123} = \\ & + \frac{ga_1^6}{3!} [16(\partial_{\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3}\phi) - 48(\partial_{\nu_1\nu_2}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3\nu_3}\phi) \\ & + 48(\partial_{\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3\nu_2\nu_3}\phi) - 16\phi_{\mu_1\mu_2\mu_3}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi) \\ & + 24(\partial_{\mu_1\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3}\phi) - 72(\partial_{\mu_1\nu_1\nu_2}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_3}\phi) \\ & + 72(\partial_{\mu_1\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_2\nu_3}\phi) - 24(\partial_{\mu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_1\nu_2\nu_3}\phi) \\ & + 12(\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3}\phi) - 36(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_3}\phi) \\ & + 36(\partial_{\mu_1\mu_2\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_2\nu_3}\phi) - 12(\partial_{\mu_1\mu_2}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_1\nu_2\nu_3}\phi) \\ & + 2(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})\phi - 6(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2}\phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_3}\phi) \\ & + 6(\partial_{\mu_1\mu_2\mu_3\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_2\nu_3}\phi) - 2(\partial_{\mu_1\mu_2\mu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_1\nu_2\nu_3}\phi)] \alpha_{\nu_1}^{(1)+}\alpha_{\nu_2}^{(1)+}\alpha_{\nu_3}^{(1)+}|0\rangle_1. \end{aligned} \quad (2.26)$$

Now let us turn to the equations with respect to  $\phi$ . It consists from the following parts:

$$l_0^{(1)}|\phi_1\rangle = -\square\phi|0\rangle_1, \quad (2.27)$$

as well as

$$\begin{aligned} & g\langle\phi_3|\langle\phi_2|e^{Y_\alpha^+}|0\rangle_{123} = \\ & + \frac{ga_1^6}{3!3!} [16\phi_{\mu_1\mu_2\mu_3}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) + 24(\partial_{\mu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) \\ & + 12(\partial_{\mu_1\mu_2\mu_3\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) + 192(\partial_{\mu_3\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3}) \\ & + 144(\partial_{\mu_1\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) + 96(\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3}\phi_{\nu_1\nu_2\nu_3}) \\ & + 36(\partial_{\mu_1\mu_2\nu_1}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) + 144(\partial_{\mu_3\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\nu_1}\phi_{\nu_1\nu_2\nu_3}) \\ & + (\partial_{\mu_1\mu_2\mu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3}) + 64(\partial_{\nu_1\nu_2\nu_3}\phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3}\phi_{\nu_1\nu_2\nu_3})] |0\rangle_1, \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} & -ga_1^2\langle C_3|\langle C_2|e^{Y_\alpha^+}|0\rangle_{123} = \frac{ga_1^6}{4} [8(\partial_{\mu_1\mu_2\nu_1\nu_2}C_{\mu_1\mu_2})C_{\nu_1\nu_2} \\ & + 32(\partial_{\nu_1}C_{\mu_1\mu_2})(\partial_{\nu_2\mu_1\mu_2}C_{\nu_1\nu_2}) + 8(\partial_{\mu_1}C_{\mu_1\mu_2})(\partial_{\mu_2\nu_1\nu_2}C_{\nu_1\nu_2}) \\ & + 16(\partial_{\nu_1\nu_2}C_{\mu_1\mu_2})(\partial_{\mu_1\mu_2}C_{\nu_1\nu_2}) + 16(\partial_{\mu_1\nu_1}C_{\mu_1\mu_2})(\partial_{\mu_2\nu_2}C_{\nu_1\nu_2}) \\ & + (\partial_{\mu_1\mu_2}C_{\mu_1\mu_2})(\partial_{\nu_1\nu_2}C_{\nu_1\nu_2})] |0\rangle_1. \end{aligned} \quad (2.29)$$

Let us turn to the equation of motion with respect to  $\langle C_1|$ . It contains parts

$$\begin{aligned} & \langle C_1| - l^{(1)}|\phi_1\rangle + l^{(1)+}|D_1\rangle = \\ & [-iC_{\nu_1\nu_2} + i\partial^{\mu_1}\phi_{\mu_1\nu_1\nu_2} - i\partial_{(\nu_1}D_{\nu_2)}] \frac{1}{2!}\alpha_{\nu_1}^{(1)+}\alpha_{\nu_2}^{(1)+}|0\rangle_1, \end{aligned} \quad (2.30)$$

$$\begin{aligned}
& -ga_1^2(\langle C_2|\langle\phi_3| + \langle C_3||\langle\phi_2\rangle)e^{Y_{\alpha^+}}|0\rangle_{123} = \\
& \frac{iga_1^6}{2} [4(\partial_{\nu_1\nu_2}C_{\mu_1\mu_2})(\partial_{\mu_1\mu_2}\phi) - 8(\partial_{\nu_1}C_{\mu_1\mu_2})(\partial_{\nu_2\mu_1\mu_2}\phi) + 4C_{\mu_1\mu_2}(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi) \\
& + 4(\partial_{\mu_1\nu_1\nu_2}C_{\mu_1\mu_2})(\partial_{\mu_2}\phi) - 8(\partial_{\mu_1\nu_1}C_{\mu_1\mu_2})(\partial_{\mu_2\nu_2}\phi) + 4(\partial_{\mu_1}C_{\mu_1\mu_2})(\partial_{\mu_2\nu_1\nu_2}\phi) \\
& + (\partial_{\mu_1\mu_2\nu_1\nu_2}C_{\mu_1\mu_2})\phi - 2(\partial_{\mu_1\mu_2\nu_1}C_{\mu_1\mu_2})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\mu_2}C_{\mu_1\mu_2})(\partial_{\nu_1\nu_2}\phi)] \alpha_{\nu_1}^{1+} \alpha_{\nu_2}^{2+} |0\rangle_{11}.
\end{aligned} \tag{2.31}$$

Finally equations of motion with respect to  $\langle D_1|$  are

$$\square D_\mu = \partial^\nu C_{\mu\nu}, \tag{2.32}$$

so the equation of motion with respect to the field  $\langle D_1|$  is not modified by nonlinear terms.

## 2.4 Gauge fixing

Let us first recall how the light-cone gauge fixing procedure can be performed for a triplet in the absence of interactions, i.e. when  $g = 0$ . To this end one can use the parameter  $\lambda_{\mu\nu}$  in (2.19) to eliminate  $\phi_{+++}, \phi_{+ij}, \phi_{++i}, \phi_{+i-}, \phi_{++-}, \phi_{+--}, (i, j = 1, \dots, \mathcal{D} - 2)$  components from the field  $\phi_{\mu_1\mu_2\mu_3}$ . After we have used the entire gauge freedom we can turn to the equations of motion. The components of the equation of motion (2.22) which contain at least one index “+” imply that the field  $C_{\mu_1\mu_2}$  is zero. In the same way the components of the equation (2.23) which contain at least one index “+” implies that the field  $D_\mu$  is zero, thus turning the equation (2.24) into an identity.

The rest of the components of the equation (2.23) implies the transversality of the fields  $\phi_{\mu_1\mu_2\mu_3}$

$$\partial^{\mu_1} \phi_{\mu_1\mu_2\mu_3} = 0, \tag{2.33}$$

the latter condition eliminating the components of the field  $\phi_{\mu_1\mu_2\mu_3}$  which contains the index “-”. Finally, the equation (2.22) gives the mass-shell condition for the longitudinal components

$$\square \phi_{i_1 i_2 i_3} = 0. \tag{2.34}$$

Let us note that this gauge fixing procedure is valid for an arbitrary number of space-time dimensions and for an arbitrary value of the spin of the triplet [31].

The modification of this procedure to the case of non-zero coupling constant is straightforward. One can check that after imposing light cone gauge on the field  $\phi_{\mu_1\mu_2\mu_3}$  the components of the nonlinear equation (2.22) which contain at least one index “+” still imply that the field  $C_{\mu_1\mu_2}$  vanishes. Therefore one ends up with the nonlinear equations

$$\begin{aligned}
& -\frac{1}{3!} \square \phi_{i_1 i_2 i_3} + \frac{ga_1^6}{3!} [16(\partial_{i_1 i_2 i_3} \phi_{j_1 j_2 j_3})(\partial_{j_1 j_2 j_3} \phi) - 48(\partial_{i_1 i_2} \phi_{j_1 j_2 j_3})(\partial_{j_1 j_2 j_3 i_3} \phi) \\
& + 48(\partial_{i_1} \phi_{j_1 j_2 j_3})(\partial_{j_1 j_2 j_3 i_2 i_3} \phi) - 16\phi_{j_1 j_2 j_3}(\partial_{j_1 j_2 j_3 i_1 i_2 i_3} \phi)] = 0,
\end{aligned} \tag{2.35}$$

and

$$-\square\phi + \frac{64ga_1^6}{3!3!}(\partial_{i_1i_2i_3}\phi_{j_1j_2j_3})(\partial_{j_1j_2j_3}\phi_{i_1i_2i_3}) = 0. \quad (2.36)$$

The results of this section can be summarized as follows. For massless higher spin fields one first constructs a free Lagrangian (the first term in (2.10)) using a nilpotent BRST charge (2.8). Then using the linear gauge transformations (the first term in (2.11)) and free equations of motion one can gauge away all auxiliary fields present in the free Lagrangian. The next step is to construct a cubic Lagrangian and to make a nonlinear deformation of the initial linear gauge transformations in such a way that the number of gauge parameters are preserved. Here we have obtained all relevant equations explicitly for the system 3 – 3 – 0 and finally demonstrated that after the total gauge fixing the number of propagating degrees of freedom, which now obey nonlinear equations of motion, is preserved.

Although these results might have been anticipated for the case of massless fields we believe that this explicit consideration is useful especially as a preparation for the case of massive fields, where the situation is completely different.

### 3 3-3-0 Vertex: Massive Fields

In this Section we consider in detail the structure of the cubic interaction and equations of motion for two massive spin 3 triplets coupled to a massive scalar. As in the case of massless higher spin fields we again have some free parameters in the BRST invariant vertex. The goal is to consider the gauge fixing procedure at the nonlinear (cubic) level and study whether the gauge invariance imposes such strong restrictions on the parameters as in the massless case.

#### 3.1 Fields and Parameters

The general line for the construction of the cubic Lagrangians for massive fields is the same as that given in Section 2.1, with a few differences to be discussed below.

The cubic Lagrangian (2.10) and gauge transformations (2.11) have the same form as for the massless fields. However, the nilpotent BRST charge  $Q^{(i)}$ , which can be obtained from the dimensional reduction from a  $(\mathcal{D} + 1)$ -dimensional massless theory, i.e. from the charge (2.8), is given by [35]–[36]\*\*

$$Q^{(i)} = c_0^{(i)}(l_0^{(i)} + m_i^2) + c^{(i)+}(l^{(i)} + m_i\alpha_D^i) + c^{(i)}(l^{(i)+} + m_i\alpha_D^{(i)+}) - c^{(i)+}c^{(i)}b_0^{(i)}, \quad i = 1, 2, 3, \quad (3.1)$$

where  $m_i$  are the masses of the fields in the  $i^{th}$  Hilbert space. An extra oscillator

$$[\alpha_D^{(i)}, \alpha_D^{(j)+}] = \delta^{ij}, \quad (3.2)$$

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\*\* Free higher spin bosonic Lagrangian theory can also be formulated on the base of BRST construction without dimensional reduction both in flat and in AdS spaces [37].

corresponds to the reduced dimension. Let us note that the mass parameter is not necessarily a constant, rather it can be a function of the spin, thus describing a Regge trajectory [36].

An off-shell cubic vertex for the massive higher spin fields with different masses was given in [9] in terms of the BRST closed forms

$$L^{(i)} = a_1 \left( \alpha^{(i)} \cdot (p^{(i+1)} - p^{(i+2)}) - c^{(i)}(b_0^{(i+2)} - b_0^{(i+1)}) - \frac{m_{i+1}^2 - m_{i+2}^2}{m_i} \alpha_D^{(i)} \right), \quad (3.3)$$

and

$$Q^{(i,i+1)} = a_2 \left( \alpha^{(i)} \cdot \alpha^{(i+1)} + \frac{\alpha_D^{(i)}}{2a_1 m_i} L^{(i+1)} - \frac{\alpha_D^{(i+1)}}{2a_1 m_{i+1}} L^{(i)} - \frac{m_i^2 + m_{i+1}^2 - m_{i+2}^2}{2m_i m_{i+1}} \alpha_D^{(i)} \alpha_D^{(i+1)} - \frac{1}{2} b^{(i)} c^{(i+1)} - \frac{1}{2} b^{(i+1)} c^{(i)} \right), \quad (3.4)$$

where  $a_1$  and  $a_2$  are arbitrary real constants. Similarly to the cubic vertex for massless higher spin fields, since the expressions (3.3) and (3.4) are separately BRST invariant, any function of these expressions is BRST invariant as well.

The higher spin functionals for the massive triplet can be deduced from the dimensional reduction of the massless triplets. In particular, for the spin 3 triplet we have

$$\begin{aligned} |\Phi_{1,2}\rangle &= \frac{1}{3!} \phi_{\mu_1 \mu_2 \mu_3}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_{\mu_2}^{(1,2)+} \alpha_{\mu_3}^{(1,2)+} |0\rangle_{1,2} + \frac{i}{2!} h_{\mu_1 \mu_2}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_{\mu_2}^{(1,2)+} \alpha_D^{(1,2)+} |0\rangle_{1,2} \\ &+ b_{\mu_1}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_D^{(1,2)+} \alpha_D^{(1,2)+} |0\rangle_{1,2} + i\varphi(x) \alpha_D^{(1,2)+} \alpha_D^{(1,2)+} \alpha_D^{(1,2)+} |0\rangle_{1,2} \\ &- \frac{i}{2!} C_{\mu_1 \mu_2}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_{\mu_2}^{(1,2)+} c_0^{(1,2)} b^{(1,2)+} |0\rangle_{1,2} \\ &- C_{\mu_1}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_D^{(1,2)+} c_0^{(1,2)} b^{(1,2)+} |0\rangle_{1,2} - iC(x) \alpha_D^{(1,2)+} \alpha_D^{(1,2)+} c_0^{(1,2)} b^{(1,2)+} |0\rangle_{1,2} \\ &+ D_{\mu_1}(x) \alpha_{\mu_1}^{(1,2)+} c^{(1,2)+} b^{(1,2)+} |0\rangle_{1,2} + iD(x) \alpha_D^{(1,2)+} c^{(1,2)+} b^{(1,2)+} |0\rangle_{1,2}, \end{aligned} \quad (3.5)$$

and

$$|\Phi_3\rangle = \phi(x) |0\rangle_3. \quad (3.6)$$

Similarly, parameters of the gauge transformations take the form

$$\begin{aligned} |\Lambda_{1,2}\rangle &= \frac{i}{2!} \lambda_{\mu_1 \mu_2}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_{\mu_2}^{(1,2)+} b^{(1,2)+} |0\rangle_{1,2} + \lambda_{\mu_1}(x) \alpha_{\mu_1}^{(1,2)+} \alpha_D^{(1,2)+} b^{(1,2)+} |0\rangle_{1,2} \\ &+ i\lambda(x) \alpha_D^{(1,2)+} \alpha_D^{(1,2)+} b^{(1,2)+} |0\rangle_{1,2}, \end{aligned} \quad (3.7)$$

and

$$|\Lambda_3\rangle = 0. \quad (3.8)$$

As we shall see below when discussing the gauge fixing procedure the fields  $h_{\mu_1 \mu_2}$ ,  $b_{\mu_1}$ ,  $\varphi$ , are Stückelberg fields, which are required for the gauge invariant description of massive fields, whereas the fields  $C_{\mu_1 \mu_2}$ ,  $C_{\mu_1}$ ,  $C$ ,  $D_{\mu_1}$  and  $D$  are auxiliary fields, similar to the ones that are present in the description of the massless triplet.

In the case of the 3 – 3 – 0 vertex the full expression is given by

$$|V\rangle = \left[ \frac{1}{3!3!}(L^{(1)+})^3(L^{(2)+})^3 + \frac{1}{2!2!}(L^{(1)+})^2(L^{(2)+})^2Q^{(12)+} + \frac{1}{2!}L^{(1)+}L^{(2)+}(Q^{(12)+})^2 + \frac{1}{3!}(Q^{(12)+})^3 \right] c_0^{(1)}c_0^{(2)}c_0^{(3)}|0\rangle_{123}, \quad (3.9)$$

where the operators  $L^{(1,2)+}$  and  $Q^{(12)+}$  are hermitian conjugate to the operators (3.3) and (3.4). We will refer to various parts of the vertex (3.9) as

$$V_6 := \frac{1}{3!3!}(L^{(1)+})^3(L^{(2)+})^3, \quad V_5 := \frac{1}{2!2!}(L^{(1)+})^2(L^{(2)+})^2Q^{(12)+}, \quad (3.10)$$

$$V_4 := \frac{1}{2!}L^{(1)+}L^{(2)+}(Q^{(12)+})^2, \quad V_3 := \frac{1}{3!}(Q^{(12)+})^3, \quad (3.11)$$

where the subscripts denote the maximal number of derivatives which will appear in the corresponding Lagrangian. It is easiest when computing the expressions for the Lagrangian and gauge transformations to consider the contributions from each of these vertices separately.

Let us note that since we are considering the identical triplets in the first and in the second Hilbert spaces, one has  $m_1 = m_2$ . Further considerable simplification occurs when the mass of the scalar field is equal to the mass of the triplet. Therefore below we will consider the situation when  $m_1 = m_2 = m_3 := m$ .

Another important point is that keeping in mind the subsequent application of the Velo-Zwanziger procedure we shall consider only the  $V_6$  and  $V_5$  parts (3.10) of the vertex. The  $V_4$  and  $V_3$  parts (3.11) of the vertex give rise to terms with fewer than two derivatives on the dynamical field and are therefore irrelevant for the causality analysis.

## 3.2 Gauge Transformations

The gauge transformations to zeroth-order in  $g$  read

$$\delta_0|\Phi_i\rangle = Q^{(i)}|\Lambda_i\rangle, \quad (3.12)$$

where  $Q^i$  are as in (3.1), which gives rise to [36]

$$\begin{aligned} \delta_0|\phi_i\rangle &= (l^{(i)+} + m_i\alpha_D^{(i)+})|\lambda_i\rangle, \\ \delta_0|C_i\rangle &= (l_0^{(i)} + m_i^2)|\lambda_i\rangle, \\ \delta_0|D_i\rangle &= (l^{(i)} + m_i\alpha_D^{(i)})|\lambda_i\rangle. \end{aligned} \quad (3.13)$$

Decomposing the equations (3.13) in terms of the component fields (3.5), (3.6)

and (3.7) for fields from the first two Hilbert spaces

$$\begin{aligned}
\delta_0 \phi_{\mu_1 \mu_2 \mu_3} &= \partial_{(\mu_1} \lambda_{\mu_2 \mu_3)}, \\
\delta_0 h_{\mu_1 \mu_2} &= -\partial_{(\mu_1} \lambda_{\mu_2)} + m \lambda_{\mu_1 \mu_2}, \\
\delta_0 b_{\mu_1} &= \partial_{\mu_1} \lambda + m \lambda_{\mu_1}, \\
\delta_0 \varphi &= m \lambda, \\
\delta_0 C_{\mu_1 \mu_2} &= (\square - m^2) \lambda_{\mu_1 \mu_2}, \\
\delta_0 C_{\mu_1} &= (\square - m^2) \lambda_{\mu_1}, \\
\delta_0 C &= (\square - m^2) \lambda, \\
\delta_0 D_{\mu_1} &= \partial_{\mu_2} \lambda_{\mu_1 \mu_2} + m \lambda_{\mu_1}, \\
\delta_0 D &= -\partial_{\mu_1} \lambda_{\mu_1} + 2m \lambda,
\end{aligned} \tag{3.14}$$

whereas for the scalar in the third Hilbert space we have

$$\delta_0 \phi = 0. \tag{3.15}$$

The contributions to the gauge transformations at first order in  $g$  are given in Appendix A.

### 3.3 Equations of motion

The full Lagrangian is given by (2.10), where we use the BRST charge (3.1). The zeroth-order contribution to the Lagrangian is

$$\begin{aligned}
L_0 &= \sum_{i=1}^2 \left[ \langle \phi_i | (l_0^{(i)} + m^2) | \phi_i \rangle - \langle \phi_i | (l^{(i)+} + m \alpha_D^{(i)+}) | C_i \rangle - \langle C_i | (l^{(i)} + m \alpha_D^{(i)}) | \phi_i \rangle \right. \\
&\quad + \langle C_i | C_i \rangle + \langle C_i | (l^{(i)+} + m \alpha_D^{(i)+}) | D_i \rangle + \langle D_i | (l^{(i)} + m \alpha_D^{(i)}) | C_i \rangle \\
&\quad \left. - \langle D_i | (l_0^{(i)} + m^2) | D_i \rangle \right] + \langle \phi_3 | (l_0^{(3)} + m_0^2) | \phi_3 \rangle.
\end{aligned} \tag{3.16}$$

The contribution to the first order in  $g$  to the Lagrangian (2.10), in the case

where the cubic interaction vertex is given by  $V_6 + V_5$ , is

$$\begin{aligned}
L_1 = & -\frac{g}{3!3!} \langle \phi_1 | \langle \phi_2 | \langle \phi_3 | (L_\alpha^{(1)+})^3 (L_\alpha^{(2)+})^3 | 0 \rangle_{123} \\
& -\frac{ga_1^2}{4} \langle C_1 | \langle C_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& -\frac{g}{4} \langle \phi_1 | \langle \phi_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} \\
& -\frac{ga_2}{8ma_1} \langle \phi_1 | \langle \phi_2 | \langle \phi_3 | \left[ \alpha_D^{(1)+} L_\alpha^{(2)+} - \alpha_D^{(2)+} L_\alpha^{(1)+} \right] (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& -ga_1^2 \langle C_1 | \langle C_2 | \langle \phi_3 | L_\alpha^{(1)+} L_\alpha^{(2)+} \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} \\
& -\frac{3ga_1a_2}{4m} \langle C_1 | \langle C_2 | \langle \phi_3 | \left[ \alpha_D^{(1)+} L_\alpha^{(2)+} - \alpha_D^{(2)+} L_\alpha^{(1)+} \right] L_\alpha^{(1)+} L_\alpha^{(2)+} | 0 \rangle_{123} \quad (3.17) \\
& -\frac{ga_1a_2}{4} \langle C_1 | \langle D_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 L_\alpha^{(2)+} | 0 \rangle_{123} + \frac{ga_1a_2}{4} \langle D_1 | \langle C_2 | \langle \phi_3 | L_\alpha^{(1)+} (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& +h.c.,
\end{aligned}$$

where  $L_\alpha^{(i)+}$  is the part of (3.3) containing only oscillators, whilst  $\hat{Q}_\alpha^{(12)+}$  is given in (A.6).

Therefore the equations of motion read:

With respect to  $\langle \phi_1 |$

$$\begin{aligned}
(l_0^{(1)} + m^2) | \phi_1 \rangle = & (l^{(1)+} + m\alpha_D^{(1)+}) | C_1 \rangle + \frac{g}{3!3!} \langle \phi_2 | \langle \phi_3 | (L_\alpha^{(1)+})^3 (L_\alpha^{(2)+})^3 | 0 \rangle_{123} \\
& + \frac{g}{4} \langle \phi_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} \\
& + \frac{ga_2}{8ma_1} \langle \phi_2 | \langle \phi_3 | \left[ \alpha_D^{(1)+} L_\alpha^{(2)+} - \alpha_D^{(2)+} L_\alpha^{(1)+} \right] (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123}. \quad (3.18)
\end{aligned}$$

With respect to  $\langle C_1 |$ :

$$\begin{aligned}
| C_1 \rangle = & (l^{(1)} + m\alpha_D^{(1)}) | \phi_1 \rangle - (l^{(1)+} + m\alpha_D^{(1)+}) | D_1 \rangle + \frac{ga_1^2}{4} \langle C_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& + ga_1^2 \langle C_2 | \langle \phi_3 | L_\alpha^{(1)+} L_\alpha^{(2)+} \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} + \frac{3ga_1a_2}{4m} \langle C_2 | \langle \phi_3 | \alpha_D^{(1)+} L_\alpha^{(1)+} (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& - \frac{3ga_1a_2}{4m} \langle C_2 | \langle \phi_3 | \alpha_D^{(2)+} (L_\alpha^{(1)+})^2 L_\alpha^{(2)+} | 0 \rangle_{123} + \frac{ga_1a_2}{4} \langle D_2 | \langle \phi_3 | (L_\alpha^{(1)+})^2 L_\alpha^{(2)+} | 0 \rangle_{123}. \quad (3.19)
\end{aligned}$$

With respect to  $\langle D_1 |$ :

$$(l_0^{(1)} + m^2) | D_1 \rangle = (l^{(1)} + m\alpha_D^{(1)}) | C_1 \rangle + \frac{ga_1a_2}{4} \langle C_2 | \langle \phi_3 | L_\alpha^{(1)+} (L_\alpha^{(2)+})^2 | 0 \rangle_{123}. \quad (3.20)$$

Let us first consider the equation of motion for  $\phi_{\mu_1\mu_2\mu_3}$ , which gives

$$\begin{aligned}
& (\square - m^2)\phi_{\mu_1\mu_2\mu_3} = \partial_{(\mu_1} C_{\mu_2\mu_3)} \\
& + \frac{ga_1^6}{3!} [8(\partial_{\mu_1\mu_2\mu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2\nu_3}\phi) + 12(\partial_{\mu_1\mu_2\mu_3\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3}\phi) \\
& + 6(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3}\phi) + (\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})\phi \\
& - 24(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_3\nu_1\nu_2\nu_3}\phi) - 36(\partial_{\mu_1\mu_2\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_3\nu_2\nu_3}\phi) \\
& - 18(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_3\nu_3}\phi) - 3(\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_3}\phi) \\
& + 24(\partial_{\mu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\mu_3\nu_1\nu_2\nu_3}\phi) + 36(\partial_{\mu_1\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\mu_3\nu_2\nu_3}\phi) \\
& + 18(\partial_{\mu_1\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\mu_3\nu_3}\phi) + 3(\partial_{\mu_1\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\mu_3}\phi) \\
& - 8\phi_{\nu_1\nu_2\nu_3}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi) - 12(\partial_{\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\mu_3\nu_2\nu_3}\phi) \\
& - 6(\partial_{\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\mu_3\nu_3}\phi) - (\partial_{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\mu_3}\phi)] \quad (3.21) \\
& + \frac{3ga_1^4a_2}{2} [4(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\mu_3})(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\mu_1\mu_2\nu_1}\phi_{\nu_1\nu_2\mu_3})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\mu_2\nu_1\nu_2}\phi_{\nu_1\nu_2\mu_3})\phi \\
& - 8(\partial_{\mu_1}\phi_{\nu_1\nu_2\mu_3})(\partial_{\mu_2\nu_1\nu_2}\phi) - 8(\partial_{\mu_1\nu_1}\phi_{\nu_1\nu_2\mu_3})(\partial_{\mu_2\nu_2}\phi) - 2(\partial_{\mu_1\nu_1\nu_2}\phi_{\nu_1\nu_2\mu_3})(\partial_{\mu_2}\phi) \\
& + 4\phi_{\nu_1\nu_2\mu_3}(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi) + 4(\partial_{\nu_1}\phi_{\nu_1\nu_2\mu_3})(\partial_{\mu_1\mu_2\nu_2}\phi) + (\partial_{\nu_1\nu_2}\phi_{\nu_1\nu_2\mu_3})(\partial_{\mu_1\mu_2}\phi)] \\
& - \frac{3ga_1^4a_2}{4m} [4(\partial_{\mu_1\mu_2\mu_3}h_{\nu_1\nu_2})(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\mu_1\mu_2\mu_3\nu_1}h_{\nu_1\nu_2})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2}h_{\nu_1\nu_2})\phi \\
& - 12(\partial_{\mu_1\mu_2}h_{\nu_1\nu_2})(\partial_{\mu_3\nu_1\nu_2}\phi) - 12(\partial_{\mu_1\mu_2\nu_1}h_{\nu_1\nu_2})(\partial_{\mu_3\nu_2}\phi) - 3(\partial_{\mu_1\mu_2\nu_1\nu_2}h_{\nu_1\nu_2})(\partial_{\mu_3}\phi) \\
& + 12(\partial_{\mu_1}h_{\nu_1\nu_2})(\partial_{\mu_2\mu_3\nu_1\nu_2}\phi) + 12(\partial_{\mu_1\nu_1}h_{\nu_1\nu_2})(\partial_{\mu_2\mu_3\nu_2}\phi) + 3(\partial_{\mu_1\nu_1\nu_2}h_{\nu_1\nu_2})(\partial_{\mu_2\mu_3}\phi) \\
& - 4h_{\nu_1\nu_2}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2}\phi) - 4(\partial_{\nu_1}h_{\nu_1\nu_2})(\partial_{\mu_1\mu_2\mu_3\nu_2}\phi) - (\partial_{\nu_1\nu_2}h_{\nu_1\nu_2})(\partial_{\mu_1\mu_2\mu_3}\phi)].
\end{aligned}$$

The equation of motion for  $h_{\mu_1\mu_2}$  is

$$\begin{aligned}
& (\square - m^2)h_{\mu_1\mu_2} = -\partial_{(\mu_1} C_{\mu_2)} + mC_{\mu_1\mu_2} \\
& + \frac{ga_1^4a_2}{4} [4(\partial_{\mu_1\mu_2}h_{\nu_1\nu_2})(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\mu_1\mu_2\nu_1}h_{\nu_1\nu_2})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\mu_2\nu_1\nu_2}h_{\nu_1\nu_2})\phi \\
& - 8(\partial_{\mu_1}h_{\nu_1\nu_2})(\partial_{\mu_2\nu_1\nu_2}\phi) - 8(\partial_{\mu_1\nu_1}h_{\nu_1\nu_2})(\partial_{\mu_2\nu_2}\phi) - 2(\partial_{\mu_1\nu_1\nu_2}h_{\nu_1\nu_2})(\partial_{\mu_2}\phi) \\
& + 4h_{\nu_1\nu_2}(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi) + 4(\partial_{\nu_1}h_{\nu_1\nu_2})(\partial_{\mu_1\mu_2\nu_2}\phi) + (\partial_{\nu_1\nu_2}h_{\nu_1\nu_2})(\partial_{\mu_1\mu_2}\phi)] \\
& + \frac{ga_1^4a_2}{4m} [8(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2\nu_3}\phi) + 12(\partial_{\mu_1\mu_2\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3}\phi) \\
& + 6(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3}\phi) + (\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})\phi \\
& - 16(\partial_{\mu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_1\nu_2\nu_3}\phi) - 24(\partial_{\mu_1\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_2\nu_3}\phi) \\
& - 12(\partial_{\mu_1\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_3}\phi) - 2(\partial_{\mu_1\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2}\phi) + 8\phi_{\nu_1\nu_2\nu_3}(\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi) \\
& + 12(\partial_{\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\nu_2\nu_3}\phi) + 6(\partial_{\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\nu_3}\phi) + (\partial_{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2}\phi)].
\end{aligned} \quad (3.22)$$

The equation of motion for  $b_{\mu_1}$  is

$$(\square - m^2)b_{\mu_1} = \partial_{\mu_1} C + mC_{\mu_1}. \quad (3.23)$$

The equation of motion for  $\varphi$  is

$$(\square - m^2)\varphi = mC. \quad (3.24)$$

The equation of motion for  $C_{\mu_1\mu_2}$  is

$$\begin{aligned} C_{\mu_1\mu_2} &= \partial_{\mu_3}\phi_{\mu_1\mu_2\mu_3} - mh_{\mu_1\mu_2} - \partial_{(\mu_1}D_{\mu_2)} \\ &- \frac{ga_1^6}{2!} [4(\partial_{\mu_1\mu_2}C_{\nu_1\nu_2})(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\mu_1\mu_2\nu_1}C_{\nu_1\nu_2})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\mu_2\nu_1\nu_2}C_{\nu_1\nu_2})\phi \\ &- 8(\partial_{\mu_1}C_{\nu_1\nu_2})(\partial_{\mu_2\nu_1\nu_2}\phi) - 8(\partial_{\mu_1\nu_1}C_{\nu_1\nu_2})(\partial_{\mu_2\nu_2}\phi) - 2(\partial_{\mu_1\nu_1\nu_2}C_{\nu_1\nu_2})(\partial_{\mu_2}\phi) \\ &+ 4C_{\nu_1\nu_2}(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi) + 4(\partial_{\nu_1}C_{\nu_1\nu_2})(\partial_{\mu_1\mu_2\nu_2}\phi) + (\partial_{\nu_1\nu_2}C_{\nu_1\nu_2})(\partial_{\mu_1\mu_2}\phi)] \\ &+ 2ga_1^4a_2 [2(\partial_{\mu_1}C_{\nu_1\mu_2})(\partial_{\nu_1}\phi) + (\partial_{\mu_1\nu_1}C_{\nu_1\mu_2})\phi - 2C_{\nu_1\mu_2}(\partial_{\mu_1\nu_1}\phi) - (\partial_{\nu_1}C_{\nu_1\mu_2})(\partial_{\mu_1}\phi)] \\ &+ \frac{3ga_1^4a_2}{2m} [2(\partial_{\mu_1\mu_2}C_{\nu_1})(\partial_{\nu_1}\phi) + (\partial_{\mu_1\mu_2\nu_1}C_{\nu_1})\phi - 4(\partial_{\mu_1}C_{\nu_1})(\partial_{\mu_2\nu_1}\phi) \\ &- 2(\partial_{\mu_1\nu_1}C_{\nu_1})(\partial_{\mu_2}\phi) + 2C_{\nu_1}(\partial_{\mu_1\mu_2\nu_1}\phi) + (\partial_{\nu_1}C_{\nu_1})(\partial_{\mu_1\mu_2}\phi)] \\ &+ \frac{ga_1^4a_2}{2} [2(\partial_{\mu_1\mu_2}D_{\nu_1})(\partial_{\nu_1}\phi) + (\partial_{\mu_1\mu_2\nu_1}D_{\nu_1})\phi - 4(\partial_{\mu_1}D_{\nu_1})(\partial_{\mu_2\nu_1}\phi) \\ &- 2(\partial_{\mu_1\nu_1}D_{\nu_1})(\partial_{\mu_2}\phi) + 2D_{\nu_1}(\partial_{\mu_1\mu_2\nu_1}\phi) + (\partial_{\nu_1}D_{\nu_1})(\partial_{\mu_1\mu_2}\phi)]. \end{aligned} \quad (3.25)$$

The equation of motion for  $C_{\mu_1}$  is

$$\begin{aligned} C_{\mu_1} &= -\partial_{\mu_2}h_{\mu_1\mu_2} - 2mb_{\mu_1} + \partial_{\mu_1}D + mD_{\mu_1} \\ &+ \frac{ga_1^4a_2}{2} [2(\partial_{\mu_1}C_{\nu_1})(\partial_{\nu_1}\phi) + (\partial_{\mu_1\nu_1}C_{\nu_1})\phi - 2C_{\nu_1}(\partial_{\mu_1\nu_1}\phi) - (\partial_{\nu_1}C_{\nu_1})(\partial_{\mu_1}\phi)]. \end{aligned} \quad (3.26)$$

The equation of motion for  $C$  is

$$C = \partial_{\mu_1}b_{\mu_1} - 3m\varphi + mD. \quad (3.27)$$

The equation of motion for  $D_{\mu_1}$  is

$$\begin{aligned} (\square - m^2)D_{\mu_1} &= \partial_{\mu_2}C_{\mu_1\mu_2} + mC_{\mu_1} \\ &- \frac{ga_1^4a_2}{4} [4(\partial_{\mu_1}C_{\nu_1\nu_2})(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\mu_1\nu_1}C_{\nu_1\nu_2})(\partial_{\nu_2}\phi) + (\partial_{\mu_1\nu_1\nu_2}C_{\nu_1\nu_2})\phi \\ &- 4C_{\nu_1\nu_2}(\partial_{\mu_1\nu_1\nu_2}\phi) - 4(\partial_{\nu_1}C_{\nu_1\nu_2})(\partial_{\mu_1\nu_2}\phi) - (\partial_{\nu_1\nu_2}C_{\nu_1\nu_2})(\partial_{\mu_1}\phi)]. \end{aligned} \quad (3.28)$$

The equation of motion for  $D$  is

$$(\square - m^2)D = -\partial_{\mu_1}C_{\mu_1} + 2mC. \quad (3.29)$$

### 3.4 Gauge fixing

In analogy with the case of massless fields considered in Section 2.4 let us discuss first the gauge fixing procedure in the absence of interactions [36].

As one can see from the equation (3.14) one can use the parameter  $\lambda$  to gauge away the field  $\varphi$ , then one can use the parameter  $\lambda_\mu$  to gauge away the field  $b_\mu$  and finally use the parameter  $\lambda_{\mu_1\mu_2}$  to gauge away the field  $h_{\mu_1\mu_2}$ .

Further, the equation (3.24) imposes  $C = 0$ , the equation (3.23) imposes  $C_\mu = 0$ , and the equation (3.25) imposes  $C_{\mu_1\mu_2} = 0$ .

We can then use (3.27) to show  $D = 0$  and (3.26) to show  $D_\mu = 0$ . With this choice, the equation of motion (3.29) for  $D$  is trivially satisfied.

Therefore one ends up with the field satisfying a mass-shell

$$(\square - m^2)\phi_{\mu_1\mu_2\mu_3} = 0, \quad (3.30)$$

and transversality (2.33) conditions.

Now let us discuss what happens in the presence of interactions, in particular when we consider the cubic vertex which consists of parts  $V_6$  and  $V_5$  given in (3.10). If one considers *only*  $V_6$ , then the procedure outlined above remains unmodified, and after the gauge fixing one obtains only the physical field  $\phi_{\mu_1\mu_2\mu_3}$  which satisfies the transversality (2.33) condition and a nonlinear equation of motion (3.21) with the parameter  $a_2$  set equal to zero. Therefore in this case the degrees of freedom are the same as for the free field.

If one considers *both*  $V_6$  and  $V_5$  then the situation is the following. After the gauge fixing, i.e. elimination of the Stückelberg fields  $h_{\mu_1\mu_2}$ ,  $b_\mu$  and  $\varphi$  via gauge transformations, the equations of motion put the fields  $C_\mu$ ,  $C$ ,  $D_\mu$  and  $D$  to be equal to zero. However the field  $C_{\mu_1\mu_2}$  which represents longitudinal (nonphysical) modes of the field  $\phi_{\mu_1\mu_2\mu_3}$  is no longer zero, but rather it satisfies the equation

$$\begin{aligned} 0 = & mC_{\mu_1\mu_2} \\ & + \frac{ga_1^4 a_2}{4m} [8(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2\nu_3}\phi) + 12(\partial_{\mu_1\mu_2\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_2\nu_3}\phi) \\ & + 6(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_3}\phi) + (\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})\phi - 16(\partial_{\mu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_1\nu_2\nu_3}\phi) \\ & - 24(\partial_{\mu_1\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_2\nu_3}\phi) - 12(\partial_{\mu_1\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_3}\phi) \\ & - 2(\partial_{\mu_1\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2}\phi) + 8\phi_{\nu_1\nu_2\nu_3}(\partial_{\mu_1\mu_2\nu_1\nu_2\nu_3}\phi) + 12(\partial_{\nu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\nu_2\nu_3}\phi) \\ & + 6(\partial_{\nu_1\nu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2\nu_3}\phi) + (\partial_{\nu_1\nu_2\nu_3}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_1\mu_2}\phi)], \end{aligned} \quad (3.31)$$

which is the equation of motion with respect to the Stückelberg field  $h_{\mu_1\mu_2}$ .

Let us consider now the scalar as a background field. As one can see from (3.31) one can consistently impose the transversality condition provided the following

constraint on the background is satisfied

$$0 = g \frac{a_1^4 a_2}{4m} [8(\partial_{\mu_1 \mu_2} \phi_{\nu_1 \nu_2 \nu_3})(\partial_{\nu_1 \nu_2 \nu_3} \phi) - 16(\partial_{\mu_1} \phi_{\nu_1 \nu_2 \nu_3})(\partial_{\mu_2 \nu_1 \nu_2 \nu_3} \phi) + 8\phi_{\nu_1 \nu_2 \nu_3}(\partial_{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3} \phi)]. \quad (3.32)$$

This constraint in turn means that one of the two conditions should be satisfied

- The triple and higher derivatives on the background field are zero.
- The mass parameter is large enough that one can ignore the term in the action (2.10) which gives rise to the right-hand side of the equation (3.32).

We shall see that after the Velo-Zwanziger like analysis of causality the first option will be ruled out.

Let us summarize the results of this section. For the case of massive higher spin fields one again starts from the gauge invariant free Lagrangian which unlike the one for massless higher spin fields contains also Stückelberg fields. The number of the parameters of gauge transformations precisely equals the number of the Stückelberg fields. After completely using the gauge freedom to eliminate the Stückelberg fields, the other auxiliary degrees of freedom are eliminated and the transversality condition on the physical field is imposed due to the free equations of motion. When performing the nonlinear (cubic) deformation of the system the number of gauge transformation parameters is preserved which again allows one to gauge away all Stückelberg fields. However since the free equations of motion are modified, then, unlike the case of massless fields, for the massive ones the transversality condition can be modified as well. As a result, the gauge invariance is not enough to determine the cubic vertex, and to preserve the true number of degrees of freedom compatible with the transversality condition one needs to impose additional restrictions on the free parameters of the interaction vertex.

## 4 Causality Analysis for 3-3-0 System

In this section we perform an analysis of causality for this system. In the case under consideration we have a coupled system of equations of motion for spin 3 and spin 0 fields. The features of this system are the higher derivatives in the interaction terms. It is clear that this system is incomplete, since one can expect an infinite system of equations involving fields of all higher spins with an infinite number of interaction vertices where the vertices can include an arbitrary number of derivatives. Any truncation will be only approximate. Therefore, it is not completely clear how causality analysis for the higher spin field theory can be carried out at all. In contrast with String Theory, which can be treated as a higher spin field model, where causality of the system of higher spin field equations of motion is stipulated by the underlying fundamental causal string equations of motion, in the higher spin field theory such

an underlying theory is unknown. Nevertheless we will try to develop the causality analysis for our truncated theory in the framework of some simplified model.

Our setting is as follows. First, we consider the scalar as a background field satisfying the free equations of motion

$$(l_0^{(3)} - m^2)|\phi_3\rangle + \mathcal{O}(g) = 0, \quad (4.1)$$

i.e.  $(\square - m^2)\phi = 0$ . This equation can be derived from the following consideration. Let us consider the coupled system of equations for spin 3 and spin 0 fields. The solution to the scalar field equation is the free field satisfying (4.1) plus the terms  $\mathcal{O}(g)$ . If we substitute such a solution into the equation for the spin 3 field we will see that the terms  $\mathcal{O}(g)$  in the solution for scalar field can be omitted.

Second, we take the some kind of low-energy approximation (i.e. the case of small  $p_\mu$ ) where we can ignore the third derivative acting on a dynamical field in comparison to the second one. Using the results from the previous section we obtain the following equation for the field  $\phi_{\mu_1\mu_2\mu_3}$

$$\begin{aligned} & (\square - m^2)\phi_{\mu_1\mu_2\mu_3} + \frac{ga_1^6}{3!} [24(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_3\nu_1\nu_2\nu_3}\phi) \\ & - 24(\partial_{\mu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\mu_3\nu_1\nu_2\nu_3}\phi) + 8\phi_{\nu_1\nu_2\nu_3}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}\phi)] \\ & + \frac{3ga_1^4a_2}{2} [-4(\partial_{\mu_1\mu_2}\phi_{\nu_1\nu_2\nu_3})(\partial_{\nu_1\nu_2}\phi) + 8(\partial_{\mu_1}\phi_{\nu_1\nu_2\nu_3})(\partial_{\mu_2\nu_1\nu_2}\phi) \\ & - 4\phi_{\nu_1\nu_2\nu_3}(\partial_{\mu_1\mu_2\nu_1\nu_2}\phi)] = 0, \end{aligned} \quad (4.2)$$

along with the transversality condition (2.33). As the first step we shall perform an analysis of the causality without taking into account the condition on the background (3.32). Then as the second step we consider the implications of the equation (3.32).

In order to perform the causality analysis in a manner analogous to [21] we consider the terms in (4.2) with two derivatives acting on the dynamical field. Let us denote derivatives of the background field  $\phi$  by

$$G_{\rho_1\dots\rho_k} := \partial_{\rho_1\dots\rho_k}\phi,$$

and define

$$\tilde{G}_{\rho_1\dots\rho_k} = p_{\rho_1} \dots p_{\rho_k}\phi(p). \quad (4.3)$$

Note that we have removed the factors of  $i$  in relation to the usual Fourier transform of  $G_{\rho_1\dots\rho_k}$ .

Hence, we are led to consider the object

$$\left[ -p^2\delta_{(\mu_1}^{\nu_1}\delta_{\mu_2}^{\nu_2}\delta_{\mu_3}^{\nu_3)} + 4ga_1^6p_{(\mu_1}p_{\mu_2}\tilde{G}_{\mu_3)}^{\nu_1\nu_2\nu_3} - 6ga_1^4a_2p_{(\mu_1}p_{\mu_2}\delta_{\mu_3)}^{\nu_3}\tilde{G}^{\nu_1\nu_2)} \right] \phi_{\nu_1\nu_2\nu_3}, \quad (4.4)$$

which we can think of as a  $N \times N$  matrix, with  $N = \binom{\mathcal{D}+2}{3}$  <sup>††</sup>, acting on the space of totally-symmetric 3-tensors which we take to have the basis

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<sup>††</sup>Recall that in general the number of independent components of a rank- $s$  totally symmetric tensor in  $\mathcal{D}$  dimensions is  $\binom{\mathcal{D}-1+s}{s}$ .

$$\{\phi_{000}, \phi_{00i}, \phi_{0ii}, \phi_{0ij}, \phi_{iii}, \phi_{iij}, \phi_{ijj}, \phi_{ijk}\},$$

for  $i = 1, \dots, \mathcal{D} - 1$  and  $i < j < k$ .

For example, the “1-1” component of (4.4) will be

$$-p^2 + 4ga_1^6(p^0)^2\tilde{G}_0^{000} - 6ga_1^4a_2(p^0)^2\tilde{G}^{00}.$$

In order to carry out the causality analysis *à la* Velo-Zwanziger we need to calculate the determinant of this matrix. Before we do so however, let us make some assumptions. In particular, we take the coupling constant  $g$  to be small, i.e. we ignore all terms of  $\mathcal{O}(g^2)$ , which is reasonable given that we are working in a perturbative framework.

Then the determinant of the matrix in (4.4) can be written as

$$D(p) = \det(-p^2\mathbb{I}_N + A) = (-p^2)^{N-1} [-(p^2) + \text{tr}(A) + \mathcal{O}(g^2)], \quad (4.5)$$

where  $A$  only contains terms proportional to the background field (and hence  $g$ ), and we have ignored higher-order contributions. In particular, we have

$$\begin{aligned} A_{(\mu_1\mu_2\mu_3)}^{(\nu_1\nu_2\nu_3)} &= \frac{4}{3}ga_1^6 \left[ p_{\mu_1}p_{\mu_2}\tilde{G}_{\mu_3}^{\nu_1\nu_2\nu_3} + p_{\mu_1}p_{\mu_3}\tilde{G}_{\mu_2}^{\nu_1\nu_2\nu_3} + p_{\mu_2}p_{\mu_3}\tilde{G}_{\mu_1}^{\nu_1\nu_2\nu_3} \right] \\ &\quad - 2ga_1^4a_2 \left[ p_{\mu_1}p_{\mu_2}\delta_{\mu_3}^{\nu_3}\tilde{G}^{\nu_1\nu_2} + p_{\mu_1}p_{\mu_3}\delta_{\mu_2}^{\nu_3}\tilde{G}^{\nu_1\nu_2} + p_{\mu_2}p_{\mu_3}\delta_{\mu_1}^{\nu_3}\tilde{G}^{\nu_1\nu_2} \right]. \end{aligned} \quad (4.6)$$

Finding the trace of this matrix then just amounts to calculating  $A_{(\mu_1\mu_2\mu_3)}^{(\mu_1\mu_2\mu_3)}$ , which is

$$\begin{aligned} \text{tr}(A) &= 4ga_1^6 p_{\mu_1}p_{\mu_2}\tilde{G}_{\mu_3}^{\mu_1\mu_2\mu_3} - 2ga_1^4a_2(\mathcal{D} + 2)p_{\mu_1}p_{\mu_2}\tilde{G}^{\mu_1\mu_2} \\ &= 2ga_1^4p_{\mu_1}p_{\mu_2} (2a_1^2p^2 - (\mathcal{D} + 2)a_2) \tilde{G}^{\mu_1\mu_2}. \end{aligned} \quad (4.7)$$

Hence, the determinant (4.5) is given by

$$D(p) = (p^2)^{N-1} \left[ p^2 + 2ga_1^4p_{\mu_1}p_{\mu_2} (2a_1^2p^2 - (\mathcal{D} + 2)a_2) \tilde{G}^{\mu_1\mu_2} \right]. \quad (4.8)$$

Assuming that the background scalar satisfies the zeroth-order equations of motion (4.1) and setting a condition on the parameters  $a_1$  and  $a_2$

$$a_2 = -\frac{2}{\mathcal{D} + 2}a_1^2m^2, \quad (4.9)$$

then (4.8) reduces to  $D(p) = (p^2)^N$ , and so we have causal propagation. On the other hand, if the equation (4.9) is not satisfied the causality is broken.

However after we performed the analysis of causality for the system we also have to take into account the condition of preservation of degrees of freedom discussed in the previous Section.

The discussion goes as follows. First recall that if we take only the  $V_6$  part of the vertex (3.10) then the number of degrees of freedom in the spin 3 triplet is unchanged without imposing any extra condition on the background. However choosing a vertex which contains only  $V_6$  implies that the constant  $a_2$  is zero. Therefore due to (4.9) the other constant  $a_1$  is zero as well and so one has no interaction. This is in contradiction with our initial assumption that  $a_1$  is non-zero. Therefore in this case the condition of correct degrees of freedom for the higher spin field and the condition of the causal propagation are not compatible with each other.

If one takes both the  $V_6$  and  $V_5$  parts in the vertex, then as we saw in the previous section one also needs to take into account the condition (3.32). Then, as we have seen, one has two options to solve it. If one requires that the triple and higher derivatives on the background scalar vanish, then the equation (4.7) implies that  $a_1^4 a_2 = 0$  which is in contradiction with the original assumption that both  $a_1, a_2 \neq 0$ . Therefore as in the previous case we have no interaction or in other words the cubic interaction allows no causal propagation for the massive fields with spin 3.

The second possibility for the preservation of degrees of freedom, i.e. the situation when the right-hand side of the equation (3.32) vanishes due to the large mass parameter<sup>‡‡</sup> in turn does not impose any further restriction on the parameters  $a_1$  and  $a_2$  apart from the one (4.9) which results from the causality analysis. Therefore one obtains that in this particular case one has causal propagation of the massive spin three field, coupled to a background scalar.

A conclusion which can be drawn from the gauge fixing procedure and from the causality analysis for massive higher spin fields is that the gauge invariance and the presence of correct number degrees of freedom in the system is not sufficient for its consistency. Rather a separate check should be performed to ensure that the propagation of a massive higher spin field is causal.

## 5 Causality Analysis for the $s - s - 0$ System

Let us consider the general spin  $s - s - 0$  system on the base of the same simplified model as in the previous section, where the spin 0 field is taken to be a background field.

In this case the cubic vertex will be a sum of the form

$$|V\rangle = \sum_{k=0}^s \frac{1}{k!((s-k)!)^2} (L^{(1+)})^{s-k} (L^{(2+)})^{s-k} (Q^{(12+)})^k c_0^1 c_0^2 c_0^3 |0\rangle_{123}, \quad (5.1)$$

whilst the higher spin functionals for the massive triplet can again be obtained from dimensional reduction of the massless spin- $s$  triplet.

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<sup>‡‡</sup>More precisely one requires that the terms suppressed by the factor  $\frac{1}{m}$  in the action can be ignored in comparison with the terms that give us the equations of motion (4.2). This condition also implies that the coupling constants and the derivatives of the background fields are of order of one or smaller.

The gauge fixing procedure for the  $s-s-0$  system follows similarly as for the spin 3-3-0 system discussed in Section 3.4, so we shall omit the full details here. Indeed, one sees again that the number of physical degrees of freedom remains unchanged when turning on the interactions provided we consider the limit of large mass  $m$ . In particular, we are left only with  $\phi_{\mu_1 \dots \mu_s}$  which satisfies a transversality condition

$$\partial^{\mu_1} \phi_{\mu_1 \dots \mu_s} = 0, \quad (5.2)$$

and a nonlinear equation of motion which, in momentum space, contains the second-derivative terms

$$-p^2 \phi_{\mu_1 \dots \mu_s} + \sum_{k=0}^{s-2} \frac{(-2a_1^2)^{s-k} a_2^k}{k!(s-k)!} \left[ p_{\mu_1} p_{\mu_2} \delta_{(\mu_{s-k+1}}^{\nu_{s-k+1}} \dots \delta_{\mu_s)}^{\nu_s} \tilde{G}_{\mu_3 \dots \mu_{s-k}}^{\nu_1 \dots \nu_{s-k}} + \text{symm} \right] \phi_{\nu_1 \dots \nu_s}. \quad (5.3)$$

Here we have already taken into account the symmetrization over indices in the  $p_{\mu_i} p_{\mu_j}$  and  $G_{\mu \dots}$  terms. For example, the relevant term for  $s=4, k=0$  would look like

$$\begin{aligned} & \frac{2g}{3} a_1^8 \left[ p_{\mu_1} p_{\mu_2} \tilde{G}_{\mu_3 \mu_4}^{\nu_1 \nu_2 \nu_3 \nu_4} + p_{\mu_1} p_{\mu_3} \tilde{G}_{\mu_2 \mu_4}^{\nu_1 \nu_2 \nu_3 \nu_4} + p_{\mu_1} p_{\mu_4} \tilde{G}_{\mu_2 \mu_3}^{\nu_1 \nu_2 \nu_3 \nu_4} \right. \\ & \left. + p_{\mu_2} p_{\mu_3} \tilde{G}_{\mu_1 \mu_4}^{\nu_1 \nu_2 \nu_3 \nu_4} + p_{\mu_2} p_{\mu_4} \tilde{G}_{\mu_1 \mu_3}^{\nu_1 \nu_2 \nu_3 \nu_4} + p_{\mu_3} p_{\mu_4} \tilde{G}_{\mu_1 \mu_2}^{\nu_1 \nu_2 \nu_3 \nu_4} \right] \phi_{\nu_1 \nu_2 \nu_3 \nu_4}. \quad (5.4) \end{aligned}$$

The causality analysis of Section 4 should then be applied to the expression (5.3), i.e. we should compute the determinant of the corresponding  $\binom{\mathcal{D}+s-1}{s} \times \binom{\mathcal{D}+s-1}{s}$  matrix acting on the vector space of symmetric  $s$ -tensors. As before, ignoring terms of  $\mathcal{O}(g^2)$ , the requirement of causal propagation corresponds exactly to the case where the trace of the first-order part of this matrix vanishes.

Indeed we find

$$\text{tr}(A) = \sum_{k=0}^{s-2} \frac{(-2a_1^2)^{s-k} a_2^k}{(s-k)!} \binom{s-k}{2} \binom{\mathcal{D}+s-1}{k} p_{\mu_1} p_{\mu_2} \tilde{G}_{\mu_3 \dots \mu_{s-k}}^{\mu_1 \dots \mu_{s-k}}. \quad (5.5)$$

Using (4.3) and the zeroth-order equation for the background scalar, we find that the condition for the vanishing of  $\text{tr}(A)$  is equivalent to finding a solution  $(x, y) \in \mathbb{R}^2$  to the homogeneous equation

$$\sum_{k=0}^{s-2} \frac{1}{(s-k)!} \binom{s-k}{2} \binom{\mathcal{D}+s-1}{k} x^{s-k-2} y^k = 0, \quad (5.6)$$

where  $x := 2a_1^2 m^2$  and  $y := a_2$ .

The idea then is that any real solution  $(x, y) \in \mathbb{R}^2$  to (5.6) corresponds to a choice of  $a_1, a_2$ , which parametrize the cubic vertex (5.1), such that the propagation of the spin  $s$  degrees of freedom is within the light-cone.

We note first that, if the only real solution to (5.6) is  $(x, y) = (0, 0)$ , then the theory can be causal only if it is free. Moreover, if  $(x, 0)$  is a solution to (5.6) then

we must have  $a_1 = 0$  and likewise for  $a_2$ . Hence, our aim is to determine whether solutions of (5.6) exist with both  $x$  and  $y$  non-zero. Since this is the case, we can divide through by  $y^{s-2}$  and reduce the problem to finding real zeroes of the degree  $s - 2$  polynomial

$$p_s^{\mathcal{D}}(z) := \sum_{k=0}^{s-2} \frac{1}{(s-k)!} \binom{s-k}{2} \binom{\mathcal{D}+s-1}{k} z^{s-k-2}, \quad (5.7)$$

in  $z = -2a_1^2 m^2 a_2^{-1}$ .

In order to solve this problem, we first rewrite (5.7) as

$$p_s^{\mathcal{D}}(z) = \frac{1}{2} \sum_{k=0}^{s-2} \binom{\mathcal{D}+s-1}{s-2-k} \frac{z^k}{k!} = \frac{1}{2} L_{s-2}^{\mathcal{D}+1}(-z), \quad (5.8)$$

where  $L_n^m(x)$  are the generalized Laguerre polynomials [38]. The  $L_n^m(x)$  are orthogonal on  $x \in [0, \infty)$  with weight function  $w(x) = e^{-x} x^m$  [38] and so, from the theory of orthogonal polynomials [39], have  $n$  real zeroes in the range  $[0, \infty)$ .

Hence, we see that for any integer spin  $s > 2$  and in any spacetime dimension  $\mathcal{D}$ , the equation  $p_s^{\mathcal{D}}(z) = 0$  will have  $s - 2$  real solutions (all with  $z < 0$ ). Each of these zeroes (which can be found numerically if necessary) correspond to a particular codimension 1 locus in  $(a_1, a_2)$  parameter space along which the cubic vertex (5.1) will give rise to causal propagation for the massive spin  $s$  field coupled to a background scalar.

## 6 Conclusions

The problem of consistency of an interacting theory which contains massive and massless higher spin fields on a flat background is quite challenging. A necessary condition for the consistency of such a theory is the presence of fields with all spins up to infinity interacting among themselves. The structure of interaction vertices is defined by the requirement of the gauge invariance, however this condition might not be sufficient and extra consistency conditions, like nontriviality of the S-matrix and locality, should be imposed [7], [17], [18].

Unfortunately a corresponding Lagrangian theory which contains an infinite number of fields and an infinite number of interacting vertices is not known yet. Therefore, it would be useful and instructive to try to analyse consistency of a truncated theory with a finite number of fields and vertices. Of course, any such consideration can be only approximate, however one can hope that understanding the structure of the truncated theory can shed some light on properties of the general theory.

In the present paper we tried to study this problem. We have analyzed the theory of massive spin 3 fields coupled to a massive scalar field using the triplet formulation for higher spin fields [31], [36] which is convenient for our analysis. We addressed two consistency tests: (i) if the gauge invariance can guarantee a propagation of the true

number of physical degrees of freedom and (ii) can the gauge invariance guarantee causal propagation? We found that the answers to both questions are in general negative.

First, we have considered a system of two identical massive spin 3 and one spin 0 fields interacting via a cubic vertex. The model is constructed in the framework of the BRST approach and is automatically gauge invariant. However, the gauge invariance does not fix the cubic vertex uniquely, which still contains some free constant parameters.

After using the gauge freedom and the equations of motion, and eliminating the auxiliary fields (including the Stückelberg ones) we found that the correct number of physical degrees of freedom is not preserved by the interaction though the theory under consideration was gauge invariant. To get the correct number of degrees of freedom we imposed the additional restrictions on the free parameters in the vertex. Only after that we obtained a gauge invariant model with the correct number of propagating degrees of freedom.

Second, we have analyzed the causality aspects for this model. The only known approach to a study of the causality in massive higher spin field theory is the Velo-Zwanziger procedure. Unfortunately this procedure is not directly applicable to the model since it requires the number of derivatives acting on a dynamical field to be at most two, or in other words the Velo-Zwanziger procedure is applicable for low-energy theories. Therefore to perform the causality analysis we have derived from the original model a simplified low-energy one. Finally we compared the constraints on the parameters of the theory obtained in the two independent procedures described above.

Let us stress once again that interacting higher spin gauge theories require higher derivatives in the vertices. In general, the number of these derivatives is infinite due to the infinite number of the fields which are present in the interaction. Therefore even a negative outcome of the Velo-Zwanziger like analysis, i.e. a possible incompatibility of causal propagation and interactions, would have been inconclusive for the case of the “entire” theory. We find however that for a certain range of coupling constants and a mass parameter, as well as under some conditions on a background field, causal propagation for a spin  $s$  triplet in a scalar field background is possible. This fact apart from some interesting implications for a low-energy theory might also have some indication for consistency of the “entire” high-energy theory as well.

The main outcome of the analysis performed in the paper is that the gauge invariance and the presence of correct degrees of freedom for the massive higher spin fields propagating on a flat background are not enough requirements for an overall consistency. Rather one has to perform some extra checks such as causality analysis which can bring about extra conditions on parameters of the theory.

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## A Gauge transformations. First order in $g$

We present here details of the contributions to the gauge transformations in Section 3 from the terms of order  $g$ .

The  $V_6$  part of the vertex gives the following contributions

$$\delta_1|\Phi_i\rangle = -g \int dc_0^{(i+1)} dc_0^{(i+2)} \left[ (\langle \Phi_{i+1} | \langle \Lambda_{i+2} | + \langle \Phi_{i+2} | \langle \Lambda_{i+1} |) V_6 c_0^{(1)} c_0^{(2)} c_0^{(3)} | 0 \rangle_{123} \right] \quad (\text{A.1})$$

For the fields in the first Hilbert space, we have

$$\delta_1|\phi_1\rangle = \frac{g}{3!2!} \langle \phi_3 | \langle \lambda_2 | (L_\alpha^{(1)+})^3 (L_\alpha^{(2)+})^2 | 0 \rangle_{123},$$

where we have split  $L^{(i)} = L_\alpha^{(i)} + L_{gh}^{(i)}$  into a part containing oscillators and a part containing ghosts. For component fields we find explicitly

$$\begin{aligned} \delta_1 \phi_{\mu_1 \mu_2 \mu_3} &\supset \frac{g a_1^6}{2!} [4(\partial_{\nu_1 \nu_2} \phi)(\partial_{\mu_1 \mu_2 \mu_3} \lambda_{\nu_1 \nu_2}) + 4(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2 \mu_3 \nu_2} \lambda_{\nu_1 \nu_2}) \\ &+ \phi(\partial_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) - 12(\partial_{\mu_1 \nu_1 \nu_2} \phi)(\partial_{\mu_2 \mu_3} \lambda_{\nu_1 \nu_2}) - 12(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2 \mu_3 \nu_2} \lambda_{\nu_1 \nu_2}) \\ &- 3(\partial_{\mu_1} \phi)(\partial_{\mu_2 \mu_3 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 12(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \phi)(\partial_{\mu_3} \lambda_{\nu_1 \nu_2}) + 12(\partial_{\mu_1 \mu_2 \nu_1} \phi)(\partial_{\mu_3 \nu_2} \lambda_{\nu_1 \nu_2}) \\ &+ 3(\partial_{\mu_1 \mu_2} \phi)(\partial_{\mu_3 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) - 4(\partial_{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2} \phi) \lambda_{\nu_1 \nu_2} - 4(\partial_{\mu_1 \mu_2 \mu_3 \nu_1} \phi)(\partial_{\nu_1} \lambda_{\nu_1 \nu_2}) \\ &- (\partial_{\mu_1 \mu_2 \mu_3} \phi)(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})]. \end{aligned} \quad (\text{A.2})$$

For the scalar field in the third Hilbert space we have

$$\delta_1|\phi_3\rangle = -\frac{g}{3!} \langle \phi_1 | \langle \lambda_2 | (L_\alpha^{(1)+})^3 (L_\alpha^{(2)+})^2 | 0 \rangle_{123} + \frac{g}{2!} \langle C_1 | \langle \lambda_2 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123}, \quad (\text{A.3})$$

from which we find

$$\begin{aligned}
\delta_1 \phi \supset & -\frac{ga_1^6}{3!} [32(\partial_{\nu_1\nu_2} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3} \lambda_{\nu_1\nu_2}) + 32(\partial_{\nu_1} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_1\mu_2\mu_3\nu_2} \lambda_{\nu_1\nu_2}) \\
& + 8\phi_{\mu_1\mu_2\mu_3}(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2} \lambda_{\nu_1\nu_2}) + 48(\partial_{\mu_1\nu_1\nu_2} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3} \lambda_{\nu_1\nu_2}) \\
& + 48(\partial_{\mu_1\nu_1} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_2} \lambda_{\nu_1\nu_2}) + 12(\partial_{\mu_1} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_2\mu_3\nu_1\nu_2} \lambda_{\nu_1\nu_2}) \\
& + 24(\partial_{\mu_1\mu_2\nu_1\nu_2} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3} \lambda_{\nu_1\nu_2}) + 24(\partial_{\mu_1\mu_2\nu_1} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_2} \lambda_{\nu_1\nu_2}) \\
& + 6(\partial_{\mu_1\mu_2} \phi_{\mu_1\mu_2\mu_3})(\partial_{\mu_3\nu_1\nu_2} \lambda_{\nu_1\nu_2}) + 4(\partial_{\mu_1\mu_2\mu_3\nu_1\nu_2} \phi_{\mu_1\mu_2\mu_3}) \lambda_{\nu_1\nu_2} \\
& + 4(\partial_{\mu_1\mu_2\mu_3\nu_1} \phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_2} \lambda_{\nu_1\nu_2}) + (\partial_{\mu_1\mu_2\mu_3} \phi_{\mu_1\mu_2\mu_3})(\partial_{\nu_1\nu_2} \lambda_{\nu_1\nu_2})] \\
& + \frac{ga_1^6}{2!} [16(\partial_{\nu_1\nu_2} C_{\mu_1\mu_2})(\partial_{\mu_1\mu_2} \lambda_{\nu_1\nu_2}) + 16(\partial_{\nu_1} C_{\mu_1\mu_2})(\partial_{\mu_1\mu_2\nu_2} \lambda_{\nu_1\nu_2}) \\
& + 4C_{\mu_1\mu_2}(\partial_{\mu_1\mu_2\nu_1\nu_2} \lambda_{\nu_1\nu_2}) + 16(\partial_{\mu_1\nu_1\nu_2} C_{\mu_1\mu_2})(\partial_{\mu_2} \lambda_{\nu_1\nu_2}) + 16(\partial_{\mu_1\nu_1} C_{\mu_1\mu_2})(\partial_{\mu_2\nu_2} \lambda_{\nu_1\nu_2}) \\
& + 4(\partial_{\mu_1} C_{\mu_1\mu_2})(\partial_{\mu_2\nu_1\nu_2} \lambda_{\nu_1\nu_2}) + 4(\partial_{\mu_1\mu_2\nu_1\nu_2} C_{\mu_1\mu_2}) \lambda_{\nu_1\nu_2} + 4(\partial_{\mu_1\mu_2\nu_1} C_{\mu_1\mu_2})(\partial_{\nu_2} \lambda_{\nu_1\nu_2}) \\
& + (\partial_{\mu_1\mu_2} C_{\mu_1\mu_2})(\partial_{\nu_1\nu_2} \lambda_{\nu_1\nu_2})].
\end{aligned} \tag{A.4}$$

In a similar way we find that for  $V_6$  the linear gauge transformations of  $|C\rangle$  and  $|D\rangle$  are unmodified.

The vertex  $V_5$  gives the following contribution to the gauge transformations

$$\delta_1 |\Phi_i\rangle = -g \int dc_0^{(i+1)} dc_0^{(i+2)} \left[ (\langle \Phi_{i+1} | \langle \Lambda_{i+2} | + \langle \Phi_{i+2} | \langle \Lambda_{i+1} |) V_5 c_0^{(1)} c_0^{(2)} c_0^{(3)} | 0 \rangle_{123} \right] \tag{A.5}$$

If we introduce for convenience the notation

$$Q^{(12)} = \hat{Q}^{(12)} + \frac{a_2}{2a_1 m_1} \alpha_D^{(1)} L^{(2)} - \frac{a_2}{2a_1 m_2} \alpha_D^{(2)} L^{(1)}, \quad \hat{Q}^{(12)} = \hat{Q}_\alpha^{(12)} + \hat{Q}_{gh.}^{(12)}, \tag{A.6}$$

then the transformations (A.5) translate into

$$\begin{aligned}
\delta_1 |\phi_1\rangle = & g \langle \phi_3 | \langle \lambda_2 | \left[ \frac{1}{2} (L_\alpha^{(1)+})^2 L_\alpha^{(2)+} \hat{Q}_\alpha^{(12)+} + \frac{3}{8m} \alpha_D^{(1)+} (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 \right. \\
& \left. - \frac{1}{4m} \alpha_D^{(2)+} (L_\alpha^{(1)+})^3 L_\alpha^{(2)+} \right] | 0 \rangle_{123},
\end{aligned} \tag{A.7}$$

$$\delta_1 |C_1\rangle = \frac{g}{8} \langle \phi_3 | \langle \lambda_2 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123}, \tag{A.8}$$

and

$$\begin{aligned}
\delta_1 |\phi_3\rangle = & -g \langle \phi_1 | \langle \lambda_2 | (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} \\
& + 2g \langle C_1 | \langle \lambda_2 | L_\alpha^{(1)+} L_\alpha^{(2)+} \hat{Q}_\alpha^{(12)+} | 0 \rangle_{123} - \frac{g}{2} \langle D_1 | \langle \lambda_2 | L_\alpha^{(1)+} (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& - \frac{3g}{4m} \langle \phi_1 | \langle \lambda_2 | \alpha_D^{(1)+} (L_\alpha^{(1)+})^2 (L_\alpha^{(2)+})^2 | 0 \rangle_{123} + \frac{3g}{2m} \langle C_1 | \langle \lambda_2 | \alpha_D^{(1)+} L_\alpha^{(1)+} (L_\alpha^{(2)+})^2 | 0 \rangle_{123} \\
& + \frac{g}{2m} \langle \phi_1 | \langle \lambda_2 | \alpha_D^{(2)+} (L_\alpha^{(1)+})^3 L_\alpha^{(2)+} | 0 \rangle_{123} - \frac{3g}{2m} \langle C_1 | \langle \lambda_2 | \alpha_D^{(2)+} (L_\alpha^{(1)+})^2 L_\alpha^{(2)+} | 0 \rangle_{123},
\end{aligned} \tag{A.9}$$

whereas the variation for  $|D_1\rangle$  is not modified. Equivalently one has

$$\begin{aligned}
\delta_1 \phi_{\mu_1 \mu_2 \mu_3} &\supset 3ga_1^4 a_2 [2(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2} \lambda_{\nu_1 \mu_3}) + \phi(\partial_{\nu_1 \mu_1 \mu_2} \lambda_{\nu_1 \mu_3}) - 4(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2} \lambda_{\nu_1 \mu_3}) \\
&- 2(\partial_{\mu_1} \phi)(\partial_{\mu_2 \nu_1} \lambda_{\nu_1 \mu_3}) + 2(\partial_{\mu_1 \mu_2 \nu_1} \phi) \lambda_{\nu_1 \mu_3} + (\partial_{\mu_1 \mu_2} \phi)(\partial_{\nu_1} \lambda_{\nu_1 \mu_3})] \\
&- \frac{3ga_1^4 a_2}{2m} [2(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2 \mu_3} \lambda_{\nu_1}) + \phi(\partial_{\mu_1 \mu_2 \mu_3 \nu_1} \lambda_{\nu_1}) - 6(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2 \mu_3} \lambda_{\nu_1}) \\
&- 3(\partial_{\mu_1} \phi)(\partial_{\mu_2 \mu_3 \nu_1} \lambda_{\nu_1}) + 6(\partial_{\mu_1 \mu_2 \nu_1} \phi)(\partial_{\mu_3} \lambda_{\nu_1}) + 3(\partial_{\mu_1 \mu_2} \phi)(\partial_{\mu_3 \nu_1} \lambda_{\nu_1}) \\
&- 2(\partial_{\mu_1 \mu_2 \mu_3 \nu_1} \phi) \lambda_{\nu_1} - (\partial_{\mu_1 \mu_2 \mu_3} \phi)(\partial_{\nu_1} \lambda_{\nu_1})], \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
\delta_1 h_{\mu_1 \mu_2} &\supset -\frac{ga_1^4 a_2}{2} [2(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2} \lambda_{\nu_1}) + \phi(\partial_{\mu_1 \mu_2 \nu_1} \lambda_{\nu_1}) - 4(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2} \lambda_{\nu_1}) \\
&- 2(\partial_{\mu_1} \phi)(\partial_{\mu_2 \nu_1} \lambda_{\nu_1}) + 2(\partial_{\mu_1 \mu_2 \nu_1} \phi) \lambda_{\nu_1} + (\partial_{\mu_1 \mu_2} \phi)(\partial_{\nu_1} \lambda_{\nu_1})] \\
&- \frac{3ga_1^4 a_2}{4m} [4(\partial_{\nu_1 \nu_2} \phi)(\partial_{\mu_1 \mu_2} \lambda_{\nu_1 \nu_2}) + 4(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) + \phi(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) \\
&- 8(\partial_{\mu_1 \nu_1 \nu_2} \phi)(\partial_{\mu_2} \lambda_{\nu_1 \nu_2}) - 8(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) - 2(\partial_{\mu_1} \phi)(\partial_{\mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) \\
&+ 4(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \phi) \lambda_{\nu_1 \nu_2} + 4(\partial_{\mu_1 \mu_2 \nu_1} \phi)(\partial_{\nu_2} \lambda_{\nu_1 \nu_2}) + (\partial_{\mu_1 \mu_2} \phi)(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})], \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
\delta_1 C_{\mu_1 \mu_2} &\supset \frac{ga_1^4 a_2}{4} [4(\partial_{\nu_1 \nu_2} \phi)(\partial_{\mu_1 \mu_2} \lambda_{\nu_1 \nu_2}) + 4(\partial_{\nu_1} \phi)(\partial_{\mu_1 \mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) \\
&+ \phi(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) - 8(\partial_{\mu_1 \nu_1 \nu_2} \phi)(\partial_{\mu_2} \lambda_{\nu_1 \nu_2}) - 8(\partial_{\mu_1 \nu_1} \phi)(\partial_{\mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) \\
&- 2(\partial_{\mu_1} \phi)(\partial_{\mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 4(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \phi) \lambda_{\nu_1 \nu_2} + 4(\partial_{\mu_1 \mu_2 \nu_1} \phi)(\partial_{\nu_2} \lambda_{\nu_1 \nu_2}) \\
&+ (\partial_{\mu_1 \mu_2} \phi)(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})], \tag{A.12}
\end{aligned}$$

and

$$\begin{aligned}
& \delta_1 \phi \supset -ga_1^4 a_2 [8(\partial_{\nu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_1 \mu_2} \lambda_{\nu_1 \mu_3}) + 4\phi_{\mu_1 \mu_2 \mu_3}(\partial_{\mu_1 \mu_2 \nu_1} \lambda_{\nu_1 \mu_3}) \\
& + 8(\partial_{\mu_1 \nu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_2} \lambda_{\nu_1 \mu_3}) + 4(\partial_{\mu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_2 \nu_1} \lambda_{\nu_1 \mu_3}) \\
& + 2(\partial_{\mu_1 \mu_2 \nu_1} \phi_{\mu_1 \mu_2 \mu_3}) \lambda_{\nu_1 \mu_3} + (\partial_{\mu_1 \mu_2} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\nu_1} \lambda_{\nu_1 \mu_3})] \\
& + \frac{ga_1^4 a_2}{2} [8(\partial_{\nu_1} h_{\mu_1 \mu_2})(\partial_{\mu_1 \mu_2} \lambda_{\nu_1}) + 4h_{\mu_1 \mu_2}(\partial_{\mu_1 \mu_2 \nu_1} \lambda_{\nu_1}) + 8(\partial_{\mu_1 \nu_1} h_{\mu_1 \mu_2})(\partial_{\mu_2} \lambda_{\nu_1}) \\
& + 4(\partial_{\mu_1} h_{\mu_1 \mu_2})(\partial_{\mu_2 \nu_1} \lambda_{\nu_1}) + 2(\partial_{\mu_1 \mu_2 \nu_1} h_{\mu_1 \mu_2}) \lambda_{\nu_1} + (\partial_{\mu_1 \mu_2} h_{\mu_1 \mu_2})(\partial_{\nu_1} \lambda_{\nu_1})] \\
& + 2ga_1^4 a_2 [4(\partial_{\nu_1} C_{\mu_1 \mu_2})(\partial_{\mu_1} \lambda_{\nu_1 \mu_2}) + 2C_{\mu_1 \mu_2}(\partial_{\mu_1 \nu_1} \lambda_{\nu_1 \mu_2}) \\
& + 2(\partial_{\mu_1 \nu_1} C_{\mu_1 \mu_2}) \lambda_{\nu_1 \mu_2} + (\partial_{\mu_1} C_{\mu_1 \mu_2})(\partial_{\nu_1} \lambda_{\nu_1 \mu_2})] \\
& + ga_1^4 a_2 [4(\partial_{\nu_1} C_{\mu_1})(\partial_{\mu_1} \lambda_{\nu_1}) + 2C_{\mu_1}(\partial_{\mu_1 \nu_1} \lambda_{\nu_1}) + 2(\partial_{\mu_1 \nu_1} C_{\mu_1}) \lambda_{\nu_1} + (\partial_{\mu_1} C_{\mu_1})(\partial_{\nu_1} \lambda_{\nu_1})] \\
& + \frac{ga_1^4 a_2}{2} [8(\partial_{\nu_1 \nu_2} D_{\mu_1})(\partial_{\mu_1} \lambda_{\nu_1 \nu_2}) + 8(\partial_{\nu_1} D_{\mu_1})(\partial_{\mu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 2D_{\mu_1}(\partial_{\mu_1 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) \\
& + 4(\partial_{\mu_1 \nu_1 \nu_2} D_{\mu_1}) \lambda_{\nu_1 \nu_2} + 4(\partial_{\mu_1 \nu_1} D_{\mu_1})(\partial_{\nu_2} \lambda_{\nu_1 \nu_2}) + (\partial_{\mu_1} D_{\mu_1})(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})] \\
& + \frac{3ga_1^4 a_2}{4m} [16(\partial_{\nu_1 \nu_2} h_{\mu_1 \mu_2})(\partial_{\mu_1 \mu_2} \lambda_{\nu_1 \nu_2}) + 16(\partial_{\nu_1} h_{\mu_1 \mu_2})(\partial_{\mu_1 \mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) \\
& + 4h_{\mu_1 \mu_2}(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 16(\partial_{\mu_1 \nu_1 \nu_2} h_{\mu_1 \mu_2})(\partial_{\mu_2} \lambda_{\nu_1 \nu_2}) + 16(\partial_{\mu_1 \nu_1} h_{\mu_1 \mu_2})(\partial_{\mu_2 \nu_2} \lambda_{\nu_1 \nu_2}) \\
& + 4(\partial_{\mu_1} h_{\mu_1 \mu_2})(\partial_{\mu_2 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 4(\partial_{\mu_1 \mu_2 \nu_1 \nu_2} h_{\mu_1 \mu_2}) \lambda_{\nu_1 \nu_2} + 4(\partial_{\mu_1 \mu_2 \nu_1} h_{\mu_1 \mu_2})(\partial_{\nu_2} \lambda_{\nu_1 \nu_2}) \\
& + (\partial_{\mu_1 \mu_2} h_{\mu_1 \mu_2})(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})] \\
& + \frac{3ga_1^4 a_2}{2m} [8(\partial_{\nu_1 \nu_2} C_{\mu_1})(\partial_{\mu_1} \lambda_{\nu_1 \nu_2}) + 8(\partial_{\nu_1} C_{\mu_1})(\partial_{\mu_1 \nu_2} \lambda_{\nu_1 \nu_2}) + 2C_{\mu_1}(\partial_{\mu_1 \nu_1 \nu_2} \lambda_{\nu_1 \nu_2}) \\
& + 4(\partial_{\mu_1 \nu_1 \nu_2} C_{\mu_1}) \lambda_{\nu_1 \nu_2} + 4(\partial_{\mu_1 \nu_1} C_{\mu_1})(\partial_{\nu_2} \lambda_{\nu_1 \nu_2}) + (\partial_{\mu_1} C_{\mu_1})(\partial_{\nu_1 \nu_2} \lambda_{\nu_1 \nu_2})] \\
& - \frac{ga_1^4 a_2}{2m} [16(\partial_{\nu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_1 \mu_2 \mu_3} \lambda_{\nu_1}) + 8\phi_{\mu_1 \mu_2 \mu_3}(\partial_{\mu_1 \mu_2 \mu_3 \nu_1} \lambda_{\nu_1}) \\
& + 24(\partial_{\mu_1 \nu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_2 \mu_3} \lambda_{\nu_1}) + 12(\partial_{\mu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_2 \mu_3 \nu_1} \lambda_{\nu_1}) \\
& + 12(\partial_{\mu_1 \mu_2 \nu_1} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_3} \lambda_{\nu_1}) + 6(\partial_{\mu_1 \mu_2} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\mu_3 \nu_1} \lambda_{\nu_1}) \\
& + 2(\partial_{\mu_1 \mu_2 \mu_3 \nu_1} \phi_{\mu_1 \mu_2 \mu_3}) \lambda_{\nu_1} + (\partial_{\mu_1 \mu_2 \mu_3} \phi_{\mu_1 \mu_2 \mu_3})(\partial_{\nu_1} \lambda_{\nu_1})] \\
& + \frac{3ga_1^4 a_2}{2m} [8(\partial_{\nu_1} C_{\mu_1 \mu_2})(\partial_{\mu_1 \mu_2} \lambda_{\nu_1}) + 4C_{\mu_1 \mu_2}(\partial_{\mu_1 \mu_2 \nu_1} \lambda_{\nu_1}) + 8(\partial_{\mu_1 \nu_1} C_{\mu_1 \mu_2})(\partial_{\mu_2} \lambda_{\nu_1}) \\
& + 4(\partial_{\mu_1} C_{\mu_1 \mu_2})(\partial_{\mu_2 \nu_1} \lambda_{\nu_1}) + 2(\partial_{\mu_1 \mu_2 \nu_1} C_{\mu_1 \mu_2}) \lambda_{\nu_1} + (\partial_{\mu_1 \mu_2} C_{\mu_1 \mu_2})(\partial_{\nu_1} \lambda_{\nu_1})]. \quad (\text{A.13})
\end{aligned}$$

## B Examples of the lower spin fields

### B.1 2 – 2 – 0 Vertex

We start with the massive spin two “triplets”

$$\begin{aligned}
|\Phi_{1,2}\rangle &= \frac{1}{2!}h_{\mu_1\mu_2}(x)\alpha_{\mu_1}^{(1,2)+}\alpha_{\mu_2}^{(1,2)+}|0\rangle_{1,2} + ib_{\mu_1}(x)\alpha_{\mu_1}^{(1,2)+}\alpha_D^{(1,2)+}|0\rangle_{1,2} \\
&+ \varphi(x)\alpha_D^{(1,2)+}\alpha_D^{(1,2)+}|0\rangle_{1,2} - iC_{\mu_1}(x)\alpha_{\mu_1}^{(1,2)+}c_0^{(1,2)}b^{(1,2)+}|0\rangle_{1,2} \\
&- C(x)\alpha_D^{(1,2)+}c_0^{(1,2)}b^{(1,2)+}|0\rangle_{1,2} + D(x)c^{(1,2)+}b^{(1,2)+}|0\rangle_{1,2}. \tag{B.1}
\end{aligned}$$

Here the fields  $b_\mu$  and  $\varphi$  are Stückelberg fields.

The cubic interaction vertices are given in terms of (3.3)–(3.4) as

$$V_4 = \frac{1}{2!2!}(L^{(1)+})^2(L^{(2)+})^2, \quad V_3 = L^{(1)+}L^{(2)+}Q^{(12)+}, \quad V_2 = \frac{1}{2!}(Q^{(12)+})^2. \tag{B.2}$$

Provided one can consistently impose the transversality condition  $\partial^{\mu_1}h_{\mu_1\mu_2} = 0$ , one can see from (5.5) that the requirement of causal propagation is equivalent to the condition

$$\frac{(-2a_1^2)^2}{2!}(-m^2) = 0, \tag{B.3}$$

which is only satisfied for  $a_1 = 0$ , and hence the vertices  $V_4$  and  $V_3$  vanish.

The equations of motion coming from the Lagrangian with the remaining vertex  $V_2$  are as follows.

With respect to  $h_{\mu_1\mu_2}$

$$\begin{aligned}
(\square - m^2)h_{\mu_1\mu_2} &= \partial_{(\mu_1}C_{\mu_2)} - ga_2^2h_{\mu_1\mu_2}\phi \\
&+ \frac{ga_2^2}{m} [(\partial_{\mu_1}b_{\mu_2})\phi - b_{\mu_2}(\partial_{\mu_1}\phi)] \\
&+ \frac{ga_2^2}{m} [(\partial_{\mu_1\mu_2}\varphi)\phi - 2(\partial_{\mu_1}\varphi)(\partial_{\mu_2}\phi) + \varphi(\partial_{\mu_1\mu_2}\phi)]. \tag{B.4}
\end{aligned}$$

With respect to  $b_{\mu_1}$

$$\begin{aligned}
(\square - m^2)b_{\mu_1} &= -\partial_{\mu_1}C + mC_{\mu_1} + \frac{ga_2^2}{2!}b_{\mu_1}\phi \\
&- \frac{ga_2^2}{2m} [2h_{\mu_1\nu_1}(\partial_{\nu_1}\phi) + (\partial_{\nu_1}h_{\mu_1\nu_1})\phi] \\
&- \frac{ga_2^2}{2m} [(\partial_{\mu_1}\varphi)\phi - \varphi(\partial_{\mu_1}\phi)] \\
&+ \frac{ga_2^2}{4m^2} [2(\partial_{\mu_1}b_{\nu_1})(\partial_{\nu_1}\phi) - 2b_{\nu_1}(\partial_{\mu_1\nu_1}\phi) + (\partial_{\mu_1\nu_1}b_{\nu_1})\phi - (\partial_{\nu_1}b_{\nu_1})(\partial_{\mu_1}\phi)]. \tag{B.5}
\end{aligned}$$

With respect to  $\varphi$

$$\begin{aligned}
(\square - m^2)\varphi &= mC - \frac{ga_2^2}{4}\varphi\phi \\
&+ \frac{ga_2^2}{4m} [2b_{\nu_1}(\partial_{\nu_1}\phi) + (\partial_{\nu_1}b_{\nu_1})\phi] \\
&+ \frac{ga_2^2}{4m^2} [4h_{\nu_1\nu_2}(\partial_{\nu_1\nu_2}\phi) + 4(\partial_{\nu_1}h_{\nu_1\nu_2})(\partial_{\nu_2}\phi) + (\partial_{\nu_1\nu_2}h_{\nu_1\nu_2})\phi].
\end{aligned} \tag{B.6}$$

With respect to  $C_{\mu_1}$

$$C_{\mu_1} = \partial_{\mu_2}h_{\mu_1\mu_2} - mb_{\mu_1} - \partial_{\mu_1}D. \tag{B.7}$$

With respect to  $C$

$$C = -\partial_{\mu_1}b_{\mu_1} - 2m\varphi + mD + \frac{ga_2^2}{4m}D\phi + \frac{ga_2^2}{4m}C\phi. \tag{B.8}$$

And finally with respect to  $D$

$$(\square - m^2)D = \partial_{\mu_1}C_{\mu_1} + mC - \frac{g}{2}D\phi - \frac{ga_2^2}{4m}C\phi. \tag{B.9}$$

At zeroth-order, the gauge transformations are given by

$$\begin{aligned}
\delta_0 h_{\mu_1\mu_2} &= \partial_{(\mu_1}\lambda_{\mu_2)}, \\
\delta_0 b_{\mu_1} &= -\partial_{\mu_1}\lambda + m\lambda_{\mu_1}, \\
\delta_0 \varphi &= m\lambda, \\
\delta_0 C_{\mu_1} &= (\square - m^2)\lambda_{\mu_1}, \\
\delta_0 C &= (\square - m^2)\lambda, \\
\delta_0 D &= \partial_{\mu_1}\lambda_{\mu_1} + m\lambda.
\end{aligned} \tag{B.10}$$

As with the  $3 - 3 - 0$  case, we can use the gauge parameters  $\lambda$  and  $\lambda_\mu$  to gauge away the fields  $b_\mu$  and  $\varphi$ .

In order that the interacting system describes the correct number of degrees of freedom for a massive spin 2 field, we must be able to consistently impose the transversality constraint  $\partial^\mu h_{\mu\nu} = 0$  along with setting the auxiliary fields  $C, C_{\mu_1}, D$  to zero. It can be easily achieved by taking a constant background field  $\phi = \langle\phi\rangle$ . The wave equation for the massive spin two field becomes

$$(\square - m^2)h_{\mu_1\mu_2} = -ga_2^2 h_{\mu_1\mu_2}\phi, \tag{B.11}$$

from which it is obvious that one has just a redefinition of the constant mass parameter.

## B.2 1 – 1 – 0 Vertex

Again we start with the massive spin-1 “triplets”

$$\begin{aligned} |\Phi_{1,2}\rangle &= +A_{\mu_1}(x)\alpha_{\mu_1}^{(1,2)+}|0\rangle_{1,2} + i\varphi(x)\alpha_D^{(1,2)+}|0\rangle_{1,2} \\ &\quad -iC(x)c_0^{(1,2)}b^{(1,2)+}|0\rangle_{1,2}. \end{aligned} \quad (\text{B.12})$$

Here the field  $\varphi$  is a Stückelberg field.

The cubic interaction vertices are given in terms of (3.3)–(3.4) as

$$V_2 = L^{(1)+}L^{(2)+}, \quad V_1 = Q^{(12)+}. \quad (\text{B.13})$$

After imposing the transversality condition  $\partial^\mu A_\mu = 0$ , neither of these vertices will give rise to terms in the equation of motion for  $A_\mu$  with two derivatives. Hence, the requirement of causality adds no additional constraints on the parameters  $a_1, a_2$ .

The equations of motion resulting from the Lagrangian with vertex  $V_2 + V_1$  are as follows.

With respect to  $A_{\mu_1}$

$$\begin{aligned} (\square - m^2)A_{\mu_1} &= \partial_{\mu_1}C \\ &+ ga_1^2 [2(\partial_{\mu_1}A_{\nu_1})(\partial_{\nu_1}\phi) + (\partial_{\mu_1\nu_1}A_{\nu_1})\phi - 2A_{\nu_1}(\partial_{\mu_1\nu_1}\phi) - (\partial_{\nu_1}A_{\nu_1})(\partial_{\mu_1}\phi)] \\ &- ga_2A_{\mu_1}\phi + \frac{ga_2}{2m} [(\partial_{\mu_1}\varphi)\phi - \varphi(\partial_{\mu_1}\phi)]. \end{aligned} \quad (\text{B.14})$$

With respect to  $\varphi$

$$(\square - m^2)\varphi = mC - ga_2\varphi\phi - \frac{ga_2}{2m} [2A_{\nu_1}(\partial_{\nu_1}\phi) + (\partial_{\nu_1}A_{\nu_1})\phi]. \quad (\text{B.15})$$

And finally with respect to  $C$

$$C = \partial_{\mu_1}A_{\mu_1} - m\varphi - ga_1^2C\phi. \quad (\text{B.16})$$

At zeroth-order, the gauge transformations are given by

$$\begin{aligned} \delta_0 A_{\mu_1} &= \partial_{\mu_1}\lambda, \\ \delta_0 \varphi &= m\lambda, \\ \delta_0 C &= (\square - m^2)\lambda. \end{aligned} \quad (\text{B.17})$$

We can use the gauge parameter  $\lambda$  to gauge away the field  $\varphi$ . Further, in order that the interacting system describes the correct number of degrees of freedom for a massive spin one field, one must be able to consistently impose the transversality constraint  $\partial^\mu A_\mu = 0$  along with setting the field  $C$  to zero. Again one can easily check that this can be satisfied for a constant background scalar field  $\phi = \langle\phi\rangle$ . The wave equation for the spin one field  $A_\mu$  is

$$(\square - m^2)A_\mu = -ga_2A_{\mu_1}\langle\phi\rangle, \quad (\text{B.18})$$

and therefore one has a simple redefinition of mass parameter, as it was for the case of the massive spin two field.

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