Approximate Minimization Using Newton's Method:

For a point u^i, $P_{app}(u) = P(u^i) + (u-u^i)^T \frac{\partial P}{\partial u} + \frac{1}{2}(u-u^i)^T H (u-u^i)$

$$\text{Newton's Method: } H(u^{i+1} - u^i) = -\frac{\partial P}{\partial u}(u^i)$$

$$\text{As } u^i \to u^*, \ \frac{\partial P}{\partial u} = 0.$$

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Numerical Linear Algebra:

We seek to solve systems:

$$Ku = f, \quad Kx = \lambda x, \quad Mu'' + Ku = 0.$$

1. Elimination: $A = L U$ — Lower Triangular, Upper Triangular
2. Orthogonalization: $A = QR$ — orthogonal columns, upper triangular
3. Singular Value Decomposition:

$$A = U \Sigma V^T$$

orthonormal rows

orthonormal columns

singular values

For a pos. def. K, $U = Q$ and $V^T = Q^T$, $\Sigma = \Lambda$.

Orthogonal Matrices:

Let $Q = [q_1 \cdots q_n]$, $q_i^T q_j = 0$ if $i \neq j$, $q_j^T q_j = 1$

$$\Rightarrow \quad Q^T Q = I.$$

For Q a square matrix; • $Q^{-1} = Q^T$.

• $\|Qx\| = \|x\|$.

Indeed, $\|Qx\|^2 = x^T Q^T Q x = x^T x = \|x\|^2$

Examples: Permutations, Rotations, Reflections

• $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$

$P^T P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$.

• $Q = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ Rotation in the 1-3 plane

• $H = \begin{bmatrix} -\cos(2\theta) & 0 & -\sin(2\theta) \\ 0 & 1 & 0 \\ -\sin(2\theta) & 0 & \cos(2\theta) \end{bmatrix} = I - 2uu^T$

for $u \perp$ to the plane of reflection, $u = (\cos\theta, 0, \sin\theta)$

<u>Orthogonalization $A = QR$:</u> Take A an $n \times n$ matrix with n linearly independent columns $a_1, \dots, a_n$.

1. Gram - Schmidt Algorithm:

Set $q_1 = \frac{a_1}{\|a_1\|}$.

Take $B_2 = a_2 - (q_1^T a_2) q_1$ s.t. $B \perp q_1$.

Set $q_2 = \frac{B}{\|B\|}$

⋮

$B_n = a_n - (q_1^T a_n) q_1 - \dots - (q_{n-1}^T q_n) q_{n-1}$,

Set $q_n = \frac{B_n}{\|B_n\|}$.

I.e. $\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$

$r_{11} = \|a_{1}\|$

2. Householder Algorithm:

$$[a_1 \quad a_2] = [q_1 \quad q_2 \quad q_3] \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix}$$

Matlab stores the QR decomposition in this form.

$$A = \begin{bmatrix} a_1 & \cdots & a_n \\ & & \end{bmatrix}$$

$$\Rightarrow u = a_1 - \alpha_1 e_1, \quad \alpha_1 = - e^{i \arg(\alpha_1)} \|a_1\|$$

$$\Rightarrow v = \frac{u}{\|u\|}$$

$$\Rightarrow Q_1 = I - 2vv^T$$

$$\Rightarrow Q_1 A = \begin{bmatrix} \alpha_1 & \cdots & \\ 0 & & \\ \vdots & A' & \\ 0 & & \end{bmatrix}$$

$\vdots$

Repeat for A'.

$$Q = Q_1^T \cdots Q_n^T.$$

Singular Value Decomposition:

For a symmetric matrix, $Ax = S\Lambda S^{-1}x$.

Otherwise, $A = U \Sigma V^T$, where Σ now contains singular values instead of eigenvalues:

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T U^T U \Sigma V^T$$
$$= V \underbrace{\Sigma^T \Sigma}_{\text{diagonal}} V^T$$

Find:
$A^T A v_i = \sigma_i^2 v_i$

We want $AV = U\Sigma$, so we need $u_i = \dfrac{A v_i}{\sigma_i}$. Let $v_1, \dots, v_r$ be the basis for $CS(A)$, $u_1, \dots, u_r$ the basis for $RS(A)$.

Reduced SVD: $A = U_{m\times r} \Sigma_{r\times r} V^T_{r\times n}$

$$= \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$$

To complete the process, add any orthonormal basis $v_{r+1}, \dots, v_n$ for the nullspace of A; $u_{r+1}, \dots, u_n$ an orthonormal basis for the nullspace of A^T.

Full SVD: $A = U_{m\times m} \Sigma_{m\times n} V^T_{n\times n} =$

$$\begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Ordering u_i, σ_i, v_i s.t. $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$.

$$A = \underbrace{u_1 \sigma_1 v_1^T}_{\text{largest}} + \dots + \underbrace{u_r \sigma_r v_r^T}_{\text{smallest}}$$

SVD: $Av_j = \sigma_j u_j$ for $j \le r$ $\quad A^T u_j = \sigma_j v_j$ for $j \le r$

$Av_j = 0$ for $j > r$ $\quad A^T u_j = 0$ for $j > r$

For a symmetric, pos. def K, $v_i = u_i$.

V is a rotation, Σ is a rescaling, U a rotation.

Start with $A^T A v_i = \sigma_i^2 v_i$

$$\Rightarrow v_i^T A^T A v_i = \sigma_i^2 v_i^T v_i \Rightarrow \|Av_i\| = \sigma_i \Rightarrow \|u_i\| = 0$$

$$\Rightarrow v_j^T A^T A v_i = \sigma_i^2 v_j^T v_i \Rightarrow (Av_j)\cdot(Av_i) = 0 \Rightarrow u_j^T \cdot u_i$$

$$\Rightarrow AA^T(Av_i) = \sigma_i^2 Av_i \Rightarrow AA^T u_i = \sigma_i^2 u_i.$$

Ex: Find the SVD for $A = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$. (Rank 1)

$$A^T A = \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix} \text{ has e.v.'s } \lambda = 100, 0;\ v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow u_1 = \frac{Av_1}{10} = \frac{1}{\sqrt{50}}\begin{bmatrix} 1 \\ 7 \end{bmatrix},\quad u_2 = \frac{1}{\sqrt{50}}\begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

$$\Rightarrow A = U \Sigma V^T = \frac{1}{\sqrt{50}}\begin{bmatrix} 1 & -7 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Pseudoinverse: $A^+ = V\Sigma^+ U^T$, $\quad A^+ u_i = \frac{v_i}{\sigma_i}$ for $i \le r$,

$A^+ u_i = 0$ for $i > r$.

The pseudoinverse has the same rank r as A.

$$\Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \ddots & & 0 \\ & & \sigma_r^{-1} & \\ 0 & & & 0 \end{bmatrix},$$ $\Sigma^+\Sigma$ as close to Id as possible.

$$AA^+ = \text{Projection onto } CS(A),$$

$$A^+A = \text{Proj. onto } RS(A).$$

Ex: The Pseudoinverse of $A = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$ is

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{50}}\begin{bmatrix} 1 & 7 \\ -7 & 1 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix}.$$

If $b \in CS(A)$, $Ax = b$ is solvable and A^+b is the shortest solution.

Condition Numbers and Norms:

The condition # of a symmetric matrix is

$$c(K) = \frac{\lambda_{max}}{\lambda_{min}}.$$

This measures the sensitivity of the system $Ku = f$ to changes. Let $Ku = f$, $K(u + \Delta u) = f + \Delta f$

$$\Rightarrow \Delta u = K^{-1}(\Delta f)$$

$$\Rightarrow \|\Delta u\| \le \frac{\|\Delta f\|}{\lambda_{min}(K)}.$$

$\lambda_{max}(K^{-1}) = \frac{1}{\lambda_{min}(K)}$. The eigenvalue λ_{min} measures the closeness of K to being singular, but there are 2 drawbacks to only using this as a measure:

- $1000 \cdot K\left(\frac{u}{1000}\right) = f$ but $\frac{\|\Delta u\|}{\|u\|}$ does not change.

We have:

Divide $\|\Delta u\| \le \frac{\|\Delta f\|}{\lambda_{min}(K)}$ by $\|u\| \ge \frac{\|f\|}{\lambda_{max}(K)}$

$$\Rightarrow \frac{\|\Delta u\|}{\|u\|} \le \frac{\lambda_{max}(K)}{\lambda_{min}(K)} \frac{\|\Delta f\|}{\|f\|}$$

$\Rightarrow$ Δu largest when Δf an eigenvector for λ_{min},
u smallest when f an eigenvector of λ_{max}.

When A not symmetric, $\|Ax\| \le \lambda_{max}(A)\|x\|$ can break down.
$\Rightarrow$ Other vectors can blow-up more than eigenvectors.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\|A\| = \frac{1+\sqrt{5}}{2}$, e.v.'s $= 1$

$\|A\|^2 = \lambda_{max}(A^TA) \approx 2.6$, $\frac{1}{\|A^{-1}\|^2} = \lambda_{min}(A^TA) \approx \frac{1}{2.6} \Rightarrow \|A\|\|A^{-1}\| \approx 2.6$

Def: Norm $\|A\| = \max_{x \ne 0} \frac{\|Ax\|}{\|x\|}$, Condition #: $c(A) = \|A\|\|A^{-1}\|$

For matrix norms, we have:

$$\|Ax\| \le \|A\|\|x\|, \quad \|AB\| \le \|A\|\|B\|, \quad \|A+B\| \le \|A\| + \|B\|.$$

$1000A$ has norm $1000\|A\|$ but $cn(1000A) = cn(A)$.

For a positive definite matrix,

$$\|K\| = \lambda_{max}(K).$$

Indeed, $K = Q\Lambda Q^T$ leaves lengths unchanged $\Rightarrow \|K\| = \|\Lambda\| = \lambda_{max}(K)$.

Similarly, $\|K^{-1}\| = \frac{1}{\lambda_{min}(K)} \Rightarrow c(K) = \frac{\lambda_{max}}{\lambda_{min}}$.

Ex: $Ax = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\frac{\|Ax\|}{\|x\|} = 2$.

$A^TA = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \sigma_1^2 = 4 \Rightarrow \|A\| = 2 =$ largest singular value.

The singular value $\sqrt{\lambda_{max}(A^TA)}$ is generally larger than $\lambda_{max}(A)$.

$$\|A\|^2 = \max \frac{\|Ax\|^2}{\|x\|^2} = \max \frac{x^TA^TAx}{x^Tx} = \lambda_{max}(A^TA) = \sigma_{max}^2.$$

$$\Rightarrow c(A) = \|A\| \, \|A^{-1}\| = \frac{\sigma_{max}}{\sigma_{min}}.$$

Ex: 1. $A = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow c(A) = \|A\| \, \|A^{-1}\|$ is at least $\sqrt{50}\,\sqrt{50} = 50$:

$Ax = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ has $\frac{\|Ax\|}{\|x\|} = \sqrt{50} \Rightarrow \|A\| \geq \sqrt{50}$

$A^{-1}x = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$ has $\frac{\|A^{-1}x\|}{\|x\|} = \sqrt{50} \Rightarrow \|A^{-1}\| \geq \sqrt{50}$

$$\Rightarrow \frac{\|Dx\|}{\|x\|} = .1\sqrt{50}, \quad \frac{\|Db\|}{\|b\|} = \frac{.1}{\sqrt{50}}$$

2. $$c(K_n) = \|K\| \, \|K^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}} = \frac{2 - 2\cos\frac{n\pi}{n+1}}{2 - 2\cos\frac{\pi}{n+1}} \approx \frac{4}{\pi^2}(n+1)^2$$

In computation, there is a $\log c$ loss to round off error.

Row Exchanges in $PA = LU$:

To determine the usefulness of an algorithm early on, sometimes it is necessary to re-order rows so that the largest pivot entry. Partial pivoting avoids multipliers in L larger than 1.

Equilibrium and the Stiffness Matrix:

Often statics come in the form of the framework:

$$\underbrace{e}_{\text{physical set-up}} = A\underbrace{u}_{\text{Displacements}}, \quad \underbrace{w}_{\text{Spring Tension}} = Ce, \quad \underbrace{f}_{\text{forces}} = \underbrace{A^T}_{\text{Balancing}} w$$

$$\Rightarrow \quad f = \underbrace{A^TCA}_{K}u$$

$K \equiv$ the stiffness matrix

Dynamics and Nonlinearity:

$Ku = f$ is the set-up for linear, static problems (equilibrium).

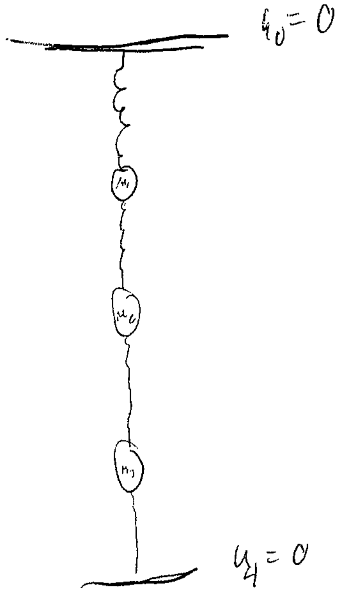
However, problems may have nonlinear components or be dynamic.

$$\Rightarrow \quad \frac{du}{dt} = Ku - f \quad \text{or} \quad M\frac{d^2u}{dt^2} + Ku = f(t)$$

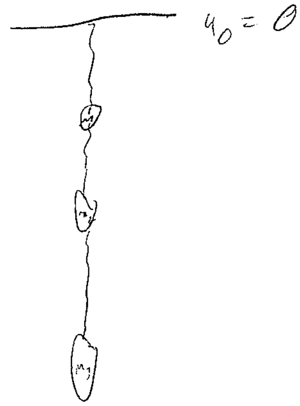
Springs:

The first application we explore will be for systems of springs.

1)



2)



Define $\vec{u} = (u_1, u_2, u_3) \equiv$ mass displacements

$\vec{w} = (w_1, w_2, w_3, w_4)$ or $(w_1, w_2, w_3) \equiv$ tension in springs

Positive Displacement is downward.

Tension positive / Compression negative.

In this case, the spring constants c_i relate the tension to the stretching: $w_i = c_i e_i$ (Force = spring constant x elongation)

The forces are produced by gravity $(m_1 g, m_2 g, m_3 g)$

To construct K, we have:

$u =$ Displacement of n masses $= (u_1, \dots, u_n)$

$e =$ Elongations of m springs $= (e_1, \dots, e_m)$

$w =$ Internal forces in m springs $= (w_1, \dots, w_m)$

$f =$ external forces on n masses $= (f_1, \dots, f_n)$

$\Rightarrow$

$$\begin{tikzcd} \boxed{u} \arrow[d, "A"'] & \boxed{f} \\ \boxed{e} \arrow[r, "C"] & \boxed{w} \arrow[u, "A^T"'] \end{tikzcd}$$

$e = Au$	A $m \times n$
$w = Ce$	C $m \times m$
$f = A^T w$	A^T $n \times m$.

First Spring: $e_1 = u_1$

Second Spring: $e_2 = u_2 - u_1$

Third Spring: $e_3 = u_3 - u_2$

Fourth Spring: $e_4 = -u_3$

Kinematic eqn

$$\Rightarrow \quad e = Au \qquad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ -u_3 \end{bmatrix}$$

Constitutive Law

Then, using Hooke's Law

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = Ce.$$

Finally, the internal forces w must balance the external forces

Static equation.

$$\Rightarrow \quad f_1 = w_1 - w_2$$
$$f_2 = w_2 - w_3 \quad \Rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$
$$f_3 = w_3 - w_4$$

$$\Rightarrow \quad f = A^T w$$

$$\Rightarrow \quad A^T C A u = f.$$

<u>Stiffness Matrix:</u> For fixed ends,

$$K = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

Note: $C = I$

$\Rightarrow K = K_3.$

Similarly, if all springs are identical, $K = K_n$ for all springs. (32)

<u>Ex:</u> $c_i = c$, $m_j = m$ $\Rightarrow$

$$u = K^{-1}f = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 1.5 \\ 2.0 \\ 1.5 \end{bmatrix}$$

stretched

$$e = Au = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\frac{mg}{c}\begin{bmatrix} 1.5 \\ 2.0 \\ 1.5 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 1.5 \\ .5 \\ -.5 \\ -1.5 \end{bmatrix}.$$

compressed

It is not generally possible to compute

$$K^{-1} = A^{-1}C^{-1}(A^T)^{-1}$$

For a fixed end / free end problem:

$$A^TCA = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & c_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

For $C = I$, $\Rightarrow$ $K = H_3$ which is T_3 with reversed ends.

<u>Ex:</u> $c_i = c$, $m_j = m$ $\Rightarrow$

$$u = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

$$\Rightarrow e = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad w = (A^T)^{-1} f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}$$

In this square case, $(A^T C A)^{-1} = (A^{-1} C^{-1} (A^T)^{-1})$.

Ex: Free-Free.

$$\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

A singular

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ represents the free shift operator that moves masses without stretching springs.

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 \\ 0 & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$$

Minimum Principles:

For a solution to $Ku = f$, we minimize the total potential energy:

$$P(u) = \frac{1}{2} u^T K u - u^T f$$

Nature seeks to minimize the total energy

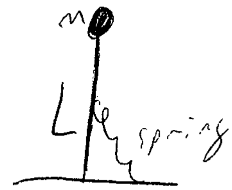
Stretching increases internal energy $\frac{1}{2} e^T C e = \frac{1}{2} u^T K u$.

Masses lose potential energy by $f^T u$

Recall: $P(u) = \frac{1}{2} u^T K u - u^T f = \frac{1}{2}(u - K^{-1} f)^T K (u - K^{-1} f) - \frac{1}{2} f^T K^{-1} f$,

$P_{min} = -\frac{1}{2} f^T K^{-1} f$ at $u = K^{-1} f$.

P_{min} negative because the masses lose potential energy when displaced.



Ex: Inverted Pendulum

$$P = \frac{1}{2} c\theta^2 + mgL\cos\theta$$

$$\frac{dP}{d\theta} = c\theta - mgL\sin\theta = 0 \Rightarrow \text{crit. points (including } \theta = 0)$$

$$\frac{d^2P}{d\theta^2} = c - mgL\cos\theta > 0 \quad \text{for stability}$$

$\Rightarrow$ stable at $\theta = 0$ if $c > mgL$

If $c < mgL$, stable at $mgL\sin\theta^* = c\theta^*$, $\theta^* \neq 0$.

$\Longrightarrow$ 2/4/2010

Oscillation Using Newton's Laws

$$Mu_{tt} = f - Ku \Rightarrow Mu_{tt} + Ku = f.$$