Fluid Models For Single Buffer Systems

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August 29, 2000

Abstract

This paper considers a stochastic fluid model of a buffer content process $\{X(t), t \geq 0\}$ that depends upon an external environment process $\{Z(t), t \geq 0\}$ as follows: whenever the environment is in state z the X process changes state at rate $\eta(z)$. The X process is restricted to stay in $[0, B]$, where $B \leq \infty$. The aim is to study the steady state distribution of the bivariate process $\{(X(t), Z(t)), t \geq 0\}$. Three main cases are considered: the environment is (i) a continuous time Markov chain (CTMC), (ii) a CTMC and white noise and (iii) an Ornstein-Uhlenbeck process. Spectral representations are obtained for the steady state distributions. Finally an extension to state dependent drift is considered, where the rate of change of the X process depends on both X and Z processes. The paper ends with some interesting open problems in this area.

*This research was partially supportd by NSF Grant No. NCR-9406823.

1 Introduction

A stochastic fluid flow system is an input-output system where the input is modeled as a continuous fluid that enters and leaves a storage device, called a buffer, according to randomly varying rates. Such models are motivated as approximations to discrete queueing models of manufacturing systems, high-speed data networks, etc. They have also been used in transportation systems, theory of dams, queueing theory, etc.

The main aim of this paper is to provide a general framework for a variety of fluid models that have been studied in literature. This is accomplished by introducing a stochastic process to model the external random environment which modulates the input and output rates of the fluid to the buffer. We then proceed to classify the models according to the stochastic nature of the random environment.

The fluid models where the external environment is a Continuous Time Markov Chain have been used in data communication networks. For example, Anick et al. [3] , Kosten [20, 21] and Kosten and Vrieze [22] treat a single buffer which receives input data from several independent sources, each source switching between on and off states according to a two-state CTMC. (This is referred to as the AMS model).The data is removed from the buffer at a fixed rate. Mitra [28, 29] considers a generalization of this model with multiple input sources and output channels, both subject to on-off switching. This work continues in Elwalid and Mitra [7] and Stern and Elwalid [37].

More recently, Asmussen [2] and Kulkarni and Karandikar [26] have studied fluid models where the environment process is a CTMC as well as white noise. These models are useful in communication networks when the effect of jitter in the channels needs to be accounted for. These are also called the second order models since they introduce the variance terms explicitly in the model.

A special limiting case of the AMS model leads to fluid models where the external environment is an Ornstein-Uhlenbeck (OU) process. Such models are studied by Simonian [35], Simonian and Virtamo [36] and Kulkarni and Rolski [27].

Although we are motivated by telecommunication applications, fluid models have been used earlier in transportation systems to model the flow of vehicles at a traffic intersection (see Newell [32]) and in dam theory (see Moran [30]). In queueing theory, the work content process can be thought of as a fluid model where the fluid arrives in instanteous quantities and leaves at a continuous rate (see Gaver and Miller [10] and Prabhu [33]).In a recent paper, Chen and Yao [6] consider a single source model with general on and off times. This can be

thought of as a fluid model with a simple two-state semi-Markovian environment process. Kella and Whitt [16] study a stochastic fluid model with Levy input.

The rest of the paper is organized on the basis of the stochastic nature of the external environment. General stability results are stated in Section 2. In Section 3 we consider the case where the environment is a CTMC and summarize the important results. Section 4 describes several applications of the model of Section 3 to high-speed networks. It includes applications to a leaky-bucket control scheme, the statistical multiplexing problem, and the admission control policies using effective bandwidth concepts. Relevant literature is cited at appropriate places.

Section 5 considers a bivariate environment process: one component is a CTMC and the other component is a white noise. In Section 6 we consider an Ornstein-Uhlenbeck process as a random environment. In Section 7 we consider the case where the input and output rates depend on the buffer content as well as the external environment. This case is useful in modeling the buffer sharing schemes in high-speed networks. The paper concludes with a brief discussion of the open problems in the area.

2 The Model

In this section we describe a general model of fluid entering and leaving a single buffer system. The input and output rates of the fluid depend on an external environment as follows: Let $Z(t)$ be the state of the environment at time t. and $X(t)$ the amount of fluid in the buffer at time t. Let $\eta(Z(t))$ be the net input rate (entry rate - exit rate) at time t. $\eta(.)$ is called the drift function. (In Section 7 we shall consider the case where the input and output rates depend on $X(t)$ as well as $Z(t)$, i.e., the drift function is given by $\eta(Z(t), X(t))$.)

When the buffer capacity is infinite, the dynamics of the buffer content process $X = \{X(t), t \geq 0\}$ is given by

$$\frac{dX(t)}{dt} = \begin{cases} \eta(Z(t)) & \text{if } X(t) > 0, \\ (\eta(Z(t)))^+ & \text{if } X(t) = 0, \end{cases} \tag{1}$$

where $(x)^+ = \max(x, 0)$. The special form at $X(t) = 0$ ensures that the X process does not become negative. When the buffer capacity is finite, say B, the dynamics is given by:

$$\frac{dX(t)}{dt} = \begin{cases} (\eta(Z(t)))^+ & \text{if } X(t) = 0, \\ \eta(Z(t)) & \text{if } 0 < X(t) < B, \\ -(\eta(Z(t)))^- & \text{if } X(t) = B, \end{cases} \tag{2}$$

where $(x)^- = \min(0, x)$. The form at $X(t) = B$ prevents the buffer content from exceeding B.

Definition. The X process is called *a fluid input-output process (or a fluid process, for short) driven by the Z process.*

In this article we shall assume the buffer capacity to be infinite unless otherwise mentioned. To solve the Equation (1) with the initial condition $X(0) = x$, we write

$$Y(t) = x + \int_0^t \eta(Z(s))ds. \tag{3}$$

Integrating both sides of Equation (1) and manipulating the result we get (see Prabhu [33])

$$X(t) = Y(t) + \int_0^t (\eta(Z(s)))^- 1_{\{X(s)=0\}} ds. \tag{4}$$

Using the standard argument of queueing theory we get

$$X(t) = Y(t) - \inf_{0 \le u \le t} (0, Y(u)) \tag{5}$$

$$= \sup_{0 \le u \le t} (Y(t), \int_u^t \eta(Z(s))ds). \tag{6}$$

The next theorem gives the result about the existence of the stationary distribution for the buffer-content process. (See Borovkov [4]).

Theorem 1 *Suppose $\{Z(t), t \ge 0\}$ is stationary and ergodic with*

$$E(\eta(Z(t))) < 0. \tag{7}$$

Then,

$$X^* = \sup_{u \le 0} \int_u^0 \eta(Z(s))ds \tag{8}$$

is an a.s. finite random variable and

$$\lim_{t \to \infty} P(X(t) > x) = P(X^* > x), \qquad x \ge 0. \tag{9}$$

In the following sections we consider fluid processes driven by various stochastic processes Z.

3 Driving Process: CTMC

In this section we study the fluid process driven by a CTMC. Let $\{Z(t), t \ge 0\}$ be an irreducible CTMC on state space $S = \{1, 2, ..., M\}$ and infinitesimal

generator matrix $Q = [q_{ij}]$. The drift function is given by $\eta(i) = d(i), i \in S$. $d(i)$ is called the drift in state i. Let

$$\begin{aligned} \pi_{ij}(t) &= P(Z(t) = j | Z(0) = i), \qquad i, j \in S, \quad (10) \\ \text{and } \pi_j &= \lim_{t \to \infty} P(Z(t) = j | Z(0) = i) \qquad i, j \in S. \quad (11) \end{aligned}$$

It is well known (see Ross [34], Kulkarni [23].) that $\Pi(t) = [\pi_{ij}(t)]$ satisfies the following equations:

$$\frac{d\Pi(t)}{dt} = \Pi(t)Q, \qquad \Pi(0) = I. \quad (12)$$

Furthermore, $\pi = (\pi_1, \pi_2, ..., \pi_M)$ is given by the unique solution to

$$\begin{aligned} \pi Q &= 0, \quad (13) \\ \sum_{i \in S} \pi_i &= 1. \quad (14) \end{aligned}$$

Transient Behavior. It is clear that $(X, Z) = \{(X(t), Z(t)), t \geq 0\}$ is a bivariate Markov process. For $0 \leq x, y < \infty$, and $i, j \in S$, let

$$F(t, x, j; y, i) = P(X(t) \leq x, Z(t) = j | X(0) = y, Z(0) = i). \quad (15)$$

The next theorem gives the forward differential equations satisfied by the transition probabilities $\{F(t, x, j; y, i)\}$. First we need the following notation:

$$\begin{aligned} F(t, x; y) &= [F(t, x, j; y, i)]_{i,j \in S}, \quad (16) \\ D &= diag(d(1), d(2), ..., d(M)). \quad (17) \end{aligned}$$

We call D the *drift matrix*.

Theorem 2 *The transition probabilities $\{F(t, x, j; y, i)\}$ satisfy the equations*

$$\frac{\partial F(t, x; y)}{\partial t} + \frac{\partial F(t, x; y)}{\partial x} D = F(t, x; y)Q, \quad (18)$$

with the boundary conditions

$$F(t, 0, j; y, i) = 0 \qquad \text{if } d(j) > 0. \quad (19)$$

If the buffer capacity is finite the additional boundary conditions are

$$F(t, B, j; y, i) = \pi_{ij}(t) \qquad \text{if } d(j) < 0. \quad (20)$$

The solution of the differential equations of Theorem 2 is a complicated task. Hence we turn our attention to the steady-state behavior.

Limiting Behavior. For the finite capacity buffer the (X, Z) process is always stable. In the case of the infinite capacity buffer, from Theorem 1, it is

clear that the buffer content process is stable, i.e., it has a limiting distribution, if

$$d = \sum_{i \in S} \pi_i d(i) < 0. \tag{21}$$

The above condition makes intuitive sense since d is the net input rate to the buffer in steady state.

We assume from now on that the above stability condition holds when the buffer capacity is infinite, and study the distribution of (X, Z). Define

$$F(x, j) = \lim_{t \to \infty} F(t, x, j; y, i), \qquad 0 \leq x < \infty, i \in S. \tag{22}$$

Note that we have implicitly assumed that the limiting distribution is independent of the initial state. Let

$$F(x) = [F(x, 1), F(x, 2), ..., F(x, M)]. \tag{23}$$

The next theorem gives the equations satisfied by $F(x)$.

Theorem 3 *$F(x)$ satisfies*

$$\frac{dF(x)}{dx} D = F(x)Q, \tag{24}$$

with the boundary conditions

$$F(0, j) = 0 \qquad \text{if } d(j) > 0. \tag{25}$$

When the buffer capacity is finite the additional boundary conditions are:

$$F(B, j) = \pi_j \qquad \text{if } d(j) < 0. \tag{26}$$

Proof: See Mitra [28, 29].

Now we develop a spectral representation of $F(x)$. Towards this end we try

$$F(x) = e^{\lambda x} \phi, \tag{27}$$

where λ is a scalar and ϕ is an M-dimensional row vector. Substituting in Equation (24) we get

$$\lambda e^{\lambda x} \phi D = e^{\lambda x} \phi Q. \tag{28}$$

This yields

$$\phi(\lambda D - Q) = 0. \tag{29}$$

Now, a non-zero vector ϕ satisfying Equation (29) exists if

$$det(\lambda D - Q) = 0. \tag{30}$$

The next theorem discusses the solutions (called the eigenvalues) λ to the above equation. See Mitra [29]. We need the following notation:

$$\begin{aligned} S_+ &= \{i \in S : d(i) > 0\}, && (31)\\ S_0 &= \{i \in S : d(i) = 0\}, && (32)\\ S_- &= \{i \in S : d(i) < 0\}, && (33)\\ M_+ &= |S_+|, && (34)\\ M_0 &= |S_0|, && (35)\\ M_- &= |S_-|. && (36) \end{aligned}$$

Theorem 4 *Equation (30) has $M_+ + M_-$ solutions (counting multiplicities) $\{\lambda_i, i = 1, 2, ..., M_+ + M_-\}$. When $d < 0$, exactly M_+ have negative real parts, 1 is zero, and $M_- - 1$ have positive real parts.*

Now consider the infinite buffer case with $d < 0$. Number the λ_i's as follows:

$$\begin{aligned} Re(\lambda_1) &\leq Re(\lambda_2) \leq ... \leq Re(\lambda_{M_+}) < \\ Re(\lambda_{M_+ + 1}) &= 0 < Re(\lambda_{M_+ + 2}) \leq ... \leq Re(\lambda_{M_+ + M_-}). \end{aligned} \quad (37)$$

We assume that all the eigenvalues $\{\lambda_i, i = 1, 2, ..., M_+ + M_-\}$ are distinct. Let $\phi_i = (\phi_{i1}, \phi_{i2}, ..., \phi_{iM})$ be the eigenvector corresponding to the eigenvalue λ_i such that the pair (λ_i, ϕ_i) satisfies Equation (29). (Note: π is the eigenvector corresponding to eigenvalue 0.) Then the general solution $F(x)$ to Equation (24) is given by

$$F(x) = \sum_{i=1}^{M_+ + M_-} a_i e^{\lambda_i x} \phi_i \quad (38)$$

where $\{a_i, i = 1, 2, ..., M_+ + M_-\}$ are scalar unknowns to be determined from the appropriate boundary conditions. The following theorem shows how this can be done. See Mitra [29].

Theorem 5 *(i) Infinite capacity buffer with $d < 0$. The a_j's are given by the solution to*

$$\begin{aligned} a_j &= 0 \qquad \text{if } Re(\lambda_j) > 0 && (39)\\ a_{M_+ + 1} &= 1, && (40)\\ \sum_{i=1}^{M_+ + 1} a_i \phi_{ij} &= 0 \qquad \text{if } j \in S_+ \ . && (41) \end{aligned}$$

(ii) Buffer with finite capacity B. The a_j's are given by the solution to;

$$\begin{aligned} \sum_{i=1}^{M_- + M_+} a_i \phi_{ij} &= 0 \qquad \text{if } j \in S_+ \ , && (42)\\ \sum_{i=1}^{M_- + M_+} a_i \phi_{ij} e^{\lambda_i B} &= \pi_j \qquad \text{if } j \in S_- \ . && (43) \end{aligned}$$

The condition in Equation (39) arises because the $F(x)$ is a bounded function of x. Note that in both cases there are as many linear equations as there are unknown a_j-s. This completes the spectral representation that we set out to obtain. We illustrate with an example.

Example 1. Suppose the input is generated by an *on-off* source. Such a source stays *on* for an $exp(\alpha)$ amount of time and stays *off* for an $exp(\beta)$ amount of time. It generates fluid at rate R when it is on and does not produce any fluid when it is off. The fluid is removed at a constant rate c from the buffer.

In this case the environment process Z is a two-state CTMC on state space $S = \{1 = \text{on}, 2 = \text{off}\}$ with the following generator matrix

$$Q = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}. \tag{44}$$

The drift matrix is given by

$$D = \begin{bmatrix} R-c & 0 \\ 0 & -c \end{bmatrix}. \tag{45}$$

Assume that the buffer capacity is infinite and that $R\beta/(\alpha+\beta) < c$ for stability. The two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \lambda = \beta/c - \alpha/(R-c) < 0$. The final solution is given by

$$F(x,1) = \frac{\beta}{(\alpha+\beta)}(1 - e^{\lambda x}), \tag{46}$$

$$F(x,2) = \frac{\alpha}{(\alpha+\beta)} - \frac{\beta(R-c)}{\alpha+\beta}e^{\lambda x}. \tag{47}$$

4 Applications

In this section we consider several applications of the fluid model of the previous section to high-speed telecommunication networks.

4.1 Congestion Control

A basic preventive congestion control strategy used in high speed networks is called the *leaky bucket* mechanism (See Elwalid and Mitra [7], Gün and Guérin [12], Guérin et al. [13], Buttó et al. [5]) that operates as follows:

Tokens enter a token pool of size M at rate γ. Each token gives permission for transmission of a single bit of information, i. e., γ is in bits/sec and M is in bits. The user generating data behaves like an on-off source as described in Example 1 above. If a token is waiting in the token pool, an arriving bit of data removes it from the token pool and enters the network. If no tokens are in the pool the incoming data waits in the data buffer of size B. When the data buffer

is full, the arriving data is lost. Similarly, if the token pool is full, the arriving tokens are lost.

Now let $Y(t)$ be the amount of data in the data buffer at time t, $Z(t)$ be the state of the source at time t, $W(t)$ be the amount of tokens in the token pool at time t. The logic of the leaky bucket implies that the data buffer and the token pool cannot be simultaneously nonempty, i.e., $W(t)Y(t) = 0$ for all t. Now define

$$X(t) = Y(t) - W(t) + M. \tag{48}$$

It can be seen that $\{X(t), t \geq 0\}$ is a fluid process on $[0, M + B]$ driven by a two-state CTMC as given in Example 1, with drifts $d(1) = \gamma$ and $d(2) = \gamma - R$. Then the limiting distribution of $X(t)$ can be written down by using the results in Example 1. The limiting distributions of $Y(t)$ and $W(t)$ can then be obtained as follows:

$$\begin{aligned}
P(Y(t) = 0) &= P(X(t) \leq M), && (49)\\
P(Y(t) > x) &= P(X(t) > x + M) \qquad \text{for } 0 \leq x < B, && (50)\\
P(Y(t) = B) &= P(X(t) = M + B), && (51)\\
P(W(t) = 0) &= P(X(t) \geq M), && (52)\\
P(W(t) > x) &= P(X(t) < M - x) \qquad \text{for } 0 \leq x < M, && (53)\\
P(W(t) = M) &= P(X(t) = 0). && (54)
\end{aligned}$$

4.2 Multiplexing in High Speed Networks

The two-state source described in Example 1 is a special case of a Markov Modulated Fluid Source (an MMFS, for short). An MMFS is described by two parameters (Q, r), where Q is the generator matrix of a CTMC on state space S and $r = [r(i)]_{i \in S}$ is a vector. When the CTMC is in state i the source produces traffic at rate $r(i)$. In high speed networks several such sources of fluid traffic are multiplexed onto a single buffer, i.e., the output from several such sources is superimposed to form a single input stream to the buffer. Such a situation can be modeled by simply constructing a large environment process Z that keeps track of the state of each source. However, the size of the state-space of the composite process undergoes a combinatorial explosion, and it makes the computation infeasible. In this section we discuss how to exploit the structure of the composite process to make the computation easier. The results here are based on Stern and Elwalid [37].

Consider the situation where K MMFS's are multiplexed onto a single infinite buffer. The k^{th} source has parameters (Q_k, r_k). Let $Z_k(t)$ be the state of the k^{th} source at time t and assume that $\{Z_k(t), t \geq 0\}$ is an irreducible CTMC on state-space $S_k = \{1, 2, ..., N_k\}$. The fluid is removed from the buffer at rate c .

Let $X(t)$ be the amount of fluid in the buffer at time t. Then it can be seen that $\{X(t), t \geq 0\}$ is a fluid process driven by the CTMC $\{Z(t) =$

$(Z_1(t), Z_2(t), ..., Z_K(t)), t \geq 0\}$. The generator matrix of the Z process is given by

$$Q = Q_1 \oplus Q_2 \oplus ... \oplus Q_K, \tag{55}$$

where $\oplus$ represents Kronecker sum. The drift matrix is given by

$$D = R_1 \oplus R_2 \oplus ... \oplus R_K - cI, \tag{56}$$

where

$$R_k = diag(r_k), \qquad \text{for } 1 \leq k \leq K. \tag{57}$$

Notice that the Q and D matrices are of size $\prod_{k=1}^{K} N_k$, which is a combinatorially large number. R_k is an N_k by N_k diagonal matrix whose ii^{th} element is $r_k(i)$. Fortunately, the problem of computing the (eigenvalue, eigenvector) pairs for these matrices can be reduced to K coupled (eigenvalue, eigenvector) problems involving smaller matrices Q_k and D_k, as explained below.

For $1 \leq k \leq K$, define

$$A_k(\lambda) = R_k - \frac{1}{\lambda} Q_k. \tag{58}$$

The main result is given in the next theorem.

Theorem 6 *A pair (λ, ϕ) satisfies Equation (29) if and only if the following equations hold:*

$$g_k(\lambda)\phi_k = \phi_k A_k(\lambda), \tag{59}$$

$$\sum_{k=1}^{K} g_k(\lambda) = c, \tag{60}$$

$$\text{and} \qquad \phi = \phi_1 \otimes \phi_2 \otimes ... \otimes \phi_K. \tag{61}$$

Example 2. Consider the multiplexing of K identical and independent on-off sources as described in Example 1. From Anick et al [3] we get

$$F(x, i) = \sum_{n=0}^{m} \phi_{ni} exp(-\lambda_n x), \tag{62}$$

where $m = K - \lceil \frac{c}{R} \rceil$, ($\lceil x \rceil$ is the largest integer less than or equal to x) and $\lambda_n, n = 0, 1, ..., m$ are the positive roots of the following $K + 1$ quadratic equations:

$$A_n \lambda^2 + B_n \lambda + C_n = 0, \qquad n = 0, 1, ..., K, \tag{63}$$

where

$$A_n = R^2(\frac{K}{2} - n)^2 - (\frac{KR}{2} - c)^2, \tag{64}$$

$$B_n = 2R(\alpha - \beta)(\frac{K}{2} - n)^2 - K(\alpha + \beta)(\frac{KR}{2} - c), \tag{65}$$

$$C_n = -(\alpha + \beta)^2\{(\frac{K}{2})^2 - (\frac{K}{2} - n)^2\}. \tag{66}$$

Furthermore, ϕ_n is the eigenvector corresponding to λ_n such that the pair (ϕ_n, λ_n) satisfies Equation (29) and can be computed easily by using the above theorem.

4.3 Effective Bandwidths

Now suppose that the network provides assurance that the incoming data will be dropped with a probability that is bounded above by a given number ϵ. Typically, $\epsilon \approx 10^{-8}$. This Quality of Service (QoS) criterion can be mathematically expressed as

$$G(B) = \lim_{t \to \infty} P(X(t) \geq B) < \epsilon. \tag{67}$$

The following theorem gives (see Elwalid and Mitra [8]) a simple yet powerful result in the asymptotic region

$$B \to \infty, \epsilon \to 0, \text{ such that } \frac{log\epsilon}{B} \to z \in (-\infty, 0]. \tag{68}$$

Theorem 7 *In the asypmtotic region in Equation (68) the QoS criterion (67) is satisfied if*

$$\sum_{k=1}^{K} g_k(z) < c. \tag{69}$$

and it is violated if

$$\sum_{k=1}^{K} g_k(z) > c. \tag{70}$$

(Note that the case $\sum_{k=1}^{K} g_k(z) = c$ is left as indeterminate bth above thteorem. In the case the QoS criterion may or may not be satisfied.) The quantity $g_k(z)$ is called the *effective bandwidth (or equivalent capacity)* of the k^{th} source, as it depends upon the Quality of Service parameter z and other source parameters (Q_k, r_k). If the sum of the effective capacities of the sources is less than the channel capacity the QoS criterion is satisfied for all the multiplexed sources. This simple additive structure provides a very useful call admission criterion. Elwalid and Mitra [8] study important properties of the effective bandwidths. The concept of effective bandwidths has its roots in the theory of large deviations and it has appeared in many other contexts. See Gibbens and Hunt [11], Guérin et al [13], Kelly [17], Kesidis and Walrand [18] etc.

5 Driving Process: CTMC + White Noise.

The fluid process studied in the previous section has piecewise deterministic sample-paths. In practice the input and output rates depend deterministically

on an external environment, but in addition, there is a small random component, called jitter, that introduces further randomness. We model this situation by a fluid process driven by a composite process $\{(Z(t), W(t)), t \geq 0\}$ where the Z component is a CTMC as described in Section 3 (with state space $S = \{1, 2, .., M\}$ and generator Q), and $\{W(t), t \geq 0\}$ is a standard white noise process. See Karlin and Taylor [15]. We consider the following drift function:

$$\eta(Z(t), W(t)) = d(Z(t)) + \sigma^2(Z(t))W(t). \tag{71}$$

One way to interpret Equations (1) and (2) is the following Ito stochastic differential equation (see Harrison [14], and Karlin and Taylor [15]):

$$dX(t) = d(Z(t)) + \sigma^2(Z(t))dB(t) \tag{72}$$

where $\{B(t), t \geq 0\}$ is the standard Brownian motion. The boundary behavior at 0 (and at B, if required) needs to be studied carefully. This fluid model is studied by Asmussen [2] and Kulkarni and Karandikar [26]. Kulkarni and Karandikar [26] study the spectral representation of the steady state distribution of the (X, Z) process, while Asmussen [2] studies the (X, Z) process via change of measure techniques. Here we concentrate on the steady-state results of Kulkarni and Karandikar [26].

When the buffer is finite, the fluid process is always stable. When it is infinite, the stability condition remains the same as in Equation (21). Assume that the process is stable and let $F(x, j), j \in S$ and $F(x)$ be as defined in Equations (22) and (23). The equations satisfied by $F(x)$ are given in the next theorem. We need the following notation:

$$\Sigma = \frac{1}{2}\begin{bmatrix} \sigma^2(1) & 0 & \cdots & 0 \\ 0 & \sigma^2(2) & \cdots & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sigma^2(M) \end{bmatrix}. \tag{73}$$

$$S_+ = \{i \in S : \sigma^2(i) > 0\}, \tag{74}$$

$$S_{0+} = \{i \in S : \sigma^2(i) = 0, d(i) > 0\}, \tag{75}$$

$$S_{00} = \{i \in S : \sigma^2(i) = 0, d(i) = 0\}, \tag{76}$$

$$S_{0-} = \{i \in S : \sigma^2(i) = 0, d(i) < 0\}. \tag{77}$$

Theorem 8 *$F(x)$ satisfies*

$$\frac{d^2F(x)}{dx^2}\Sigma - \frac{dF(x)}{dx}D + F(x)Q = 0 \tag{78}$$

with the following boundary conditions:

$$F(0, i) = 0, \qquad \text{for } i \in S_+ \cup S_{0+}. \tag{79}$$

If the buffer content, B, is finite it satisfies the following additional boundary conditions:

$$F(B,i) = \pi(i), \qquad \text{for } i \in S_+ \cup S_{0-}. \tag{80}$$

As in the previous section we derive a spectral representation for $F(x)$. Assume that $F(x)$ is as given in Equation (27), and substitute it into Equation (78). We see that (λ, ϕ) is a valid (eigenvalue, eigenvector) combination if

$$det(\lambda^2\Sigma - \lambda D + Q) = 0, \tag{81}$$

$$\phi(\lambda^2\Sigma - \lambda D + Q) = 0. \tag{82}$$

The next theorem describes the nature of the solutions (λ, ϕ) to the Equations (81) and (82). We use the following notation: $M_+ = |S_+|, M_{0+} = |S_{0+}|, M_{00} = |S_{00}|$ and $M_{0-} = |S_{0-}|$.

Theorem 9 *Equation (82) has $2M_+ + M_{0+} + M_{0-}$ solutions (counting multiplicities). When $d < 0$, exactly $M_+ + M_{0-} - 1$ have positive real parts, 1 is zero and $M_+ + M_{0+}$ have negative real parts.*

Now assume that all the eigenvalues $\{\lambda_i, i = 1, 2, ..., 2M_+ + M_{0+} + M_{0-}\}$ are distinct and arranged in ascending order of their real parts. Let ϕ_i be the eigenvector that satisfies Equation (82) for $\lambda = \lambda_i$. Then the general solution $F(x)$ to Equation (78) is given by

$$F(x) = \sum_{i=1}^{2M_+ + M_{0+} + M_{0-}} a_i e^{\lambda_i x} \phi_i, \tag{83}$$

where $\{a_i, i = 1, 2, ..., 2M_+ M_{0+} + M_{0-}\}$ are scalar unknowns to be determined from the appropriate boundary conditions. The following theorem is analogous to Theorem 5.

Theorem 10 *(i) Infinite capacity buffer with $d < 0$. The a_j's are given by the solution to*

$$a_j = 0 \qquad \text{if } Re(\lambda_j) > 0, \tag{84}$$

$$a_j = 1 \qquad \text{if } \lambda_j = 0, \tag{85}$$

$$\sum_{i=1}^{M_+ + M_{0+}} a_i \phi_{ij} = 0 \qquad \text{if } j \in S_+ \cup S_{0+}\ . \tag{86}$$

(ii) Buffer with finite capacity B. The a_j's are given by the solution to

$$\sum_{i=1}^{2M_+ + M_{0+} + M_{0-}} a_i \phi_{ij} = 0 \qquad \text{if } j \in S_+ \cup S_{0+}\ , \tag{87}$$

$$\sum_{i=1}^{2M_+ + M_{0+} + M_{0-}} a_i \phi_{ij} e^{\lambda_i B} = \pi_j \qquad \text{if } j \in S_+ \cup S_{0-}\ . \tag{88}$$

Example 3. Consider the extreme case of $M = 1$. Thus the Z process does not change state and has the generator $Q = [0]$. Let $\sigma^2(1) = \sigma^2$ and $d(1) = d$. Then the $\{X(t), t \geq 0\}$ process reduces to a standard Brownian motion on $[0, B]$ with reflection at 0 and B. Equation (81) becomes

$$\frac{1}{2}\sigma^2\lambda^2 - d\lambda = 0.$$

Hence we get $\lambda_1 = 0, \lambda_2 = 2d/\sigma^2$. If $B = \infty$ and $d < 0$ we get

$$F(x, 1) = 1 - exp\{\frac{2d}{\sigma^2}x\} \qquad \text{for } x \geq 0.$$

If B is finite we get

$$F(x, 1) = \frac{1 - exp\{\frac{2d}{\sigma^2}x\}}{1 - exp\{\frac{2d}{\sigma^2}B\}} \qquad \text{for } 0 \leq x \leq B.$$

These results match with known distributions. See Harrison [14] and Karlin and Taylor [15].

It can easily be seen that the results of this section reduce to those of the previous section if we set $\sigma^2(i) = 0$ for all $i \in S$.

6 Driving Process: Ornstein-Uhlenbeck Process

Consider the multiplexing of K on-off sources of Example 2. Suppose $R = R(K)$ goes to zero and $c = c(K)$ goes to ∞ as $K \to \infty$ in such a way that

$$R(K)\sqrt{Kf(1-f)} \quad \to \quad r, \tag{89}$$

$$c(K) - KfR(K) \quad \to \quad c, \tag{90}$$

where $f = \alpha/(\alpha + \beta)$. Under this asymptotic behavior, the fluid process of Example 2 converges to the fluid process driven by an Ornstein Uhlenbeck (OU) process with drift parameter $-(\alpha + \beta)z$ and variance parameter $2(\alpha + \beta)$. The drift function for this limiting fluid process is given by

$$\eta(z) = rz - c. \tag{91}$$

(See Kulkarni and Rolski [27], Simonian [35], Simonian and Virtamo [36].)

This motivates the study of a fluid process driven by a general Ornstein-Uhlenbeck (OU) process. Thus we assume that $\{Z(t), t \geq 0\}$ is an OU process, i.e., it is a diffusion process on $(-\infty, \infty)$ with drift parameter $\mu(b - z)$ and variance parameter σ^2, where μ and σ^2 are non-negative constants. See Karlin and Taylor [15]. We consider the drift function given in Equation (91) with $r =$

1, without loss of generality. Now define the following transformed processes:

$$X'(t) = \frac{\mu X(t/\mu)}{\sigma/\sqrt{2\mu}}, \tag{92}$$

$$Z'(t) = \frac{Z(t/\mu) - b}{\sigma/\sqrt{2\mu}} \tag{93}$$

Then the transformed process $\{Z'(t), t \geq 0\}$ is an OU process with drift $-z$ and variance parameter 2. The process $\{X'(t), t \geq 0\}$ is a fluid process driven by Z' with the following drift function:

$$\eta(z') = z' - \gamma, \tag{94}$$

where $\gamma = (c-b)/(\sigma/\sqrt{2\mu})$. From now on we omit the primes for clarity and consider this normalized (X, Z) process with a single parameter γ.

Next we study the stationary distribution of the bivariate process (X, Z). Theorem 1 implies that the process is stable if $\gamma > 0$. We assume this to be the case from now on. Now, in steady-state, the X process has a mass at zero whenever the Z process is below γ. Hence the bivariate process has an absolutely continuous density $f(x,z)$ on $S = \{(x,z)|x > 0, -\infty < z < \infty\} \cup \{(x,z)|x = 0, z > \gamma\}$, and an absolutely continuous density $f_0(z)$ on $S_0 = \{z < \gamma\}$. The next theorem gives the equations satisfied by them:

Theorem 11 *The densities $f(x,z)$ and $f_0(z)$ satisfy the following equations*

$$\frac{\partial^2 f(x,z)}{\partial z^2} + \frac{\partial}{\partial z}(z f(x,z)) = (z-\gamma)\frac{\partial f(x,z)}{\partial x}, \qquad (x,z) \in S, \tag{95}$$

$$\frac{d^2 f_0(z)}{dz^2} + \frac{d}{dz}(z f_0(z)) = (z-\gamma) f(0,z), \qquad z \in S_0, \tag{96}$$

where $f(0,z) = \lim_{x \to 0} f(x,z)$.

The solution to the above equations is given in the next theorem. First we need the following notation:

$$\omega_k = \frac{1}{2}(\sqrt{\gamma^2 + 4k} + \gamma) \qquad k \geq 0, \tag{97}$$

$$H_k(z) = (-1)^k exp(z^2/2)\frac{d^k}{dz^k} exp(-z^2/2) \qquad k \geq 0 \tag{98}$$

$$g_k(z) = exp(-\omega_k^2/2) exp(-(z-\omega_k)^2/2) H_k(\frac{z - 2\omega_k}{\sqrt{2}}) \qquad k \geq 0. \tag{99}$$

The $H_k(z)$ functions defined above are the standard Hermite polynomials. (See Andrews [1].) With this notation we have

Theorem 12 *(Knessl and Morrison [19]) The densities $f(x,z)$ and $f_0(z)$ are given by:*

$$f(x,z) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} a_k\omega_k exp(-\omega_k x)g_k(z), \qquad (x,z)\in S \tag{100}$$

$$f_0(z) = \frac{1}{\sqrt{2\pi}}exp(-z^2/2) - \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty} a_k g_k(z), \qquad z\in S_0. \tag{101}$$

The constants $a_k, k\geq 0$ are given by

$$a_0 = exp(\gamma\zeta(1/2))\prod_{m=1}^{\infty}\frac{\omega_m}{\omega_m-\gamma}exp(-\gamma/m), \tag{102}$$

$$a_k = 2^{-k/2}\frac{\gamma}{\omega_k-\gamma}exp((\zeta(1/2)-1/\sqrt{k})\omega_k + (k\psi-1)/2) \cdot \prod_{m=1,m\neq k}^{\infty}\frac{\omega_m}{\omega_m-\omega_k}exp(-\omega_k/\sqrt{m}-k/(2m)) \qquad k\geq 1, \tag{103}$$

where ψ is Euler's constant and $\zeta(\cdot)$ is Riemann's zeta function.

One consequence of the above theorem is the following asymptotic result when x is large:

$$\lim_{t\to\infty} P(X(t)>x) \sim e^{-\gamma x}. \tag{104}$$

This result has been improved in Kulkarni and Rolski [27] who prove the following bound for all $x\geq 0$, using change of measure techniques:

$$\lim_{t\to\infty} P(X(t)>x) \leq e^{-\gamma x}exp(-\gamma^2/2). \tag{105}$$

7 State-Dependent Drift

Here we consider a further generalization of the basic model of Section 2: when the external environment is z and the buffer content is x, the net input to the buffer is given by $\eta(z,x)$. There is no general theory for such a case. The stability condition for the infinite buffer case can be intuitively seen to be the following:

$$\limsup_{x\to\infty} E(\eta(Z,x)) < 0, \tag{106}$$

where Z has the steady state distribution of $\{Z(t), t\geq 0\}$. We describe below one case for which explicit results are available.

The buffer capacity is B. The given J thresholds are $0 = B_0 < B_1 < B_2 < ... < B_J = B$. The environment process is a CTMC on state-space

$S = \{1, 2, ..., M\}$ with rate matrix Q and steady state distribution π. The drift function is a step function of x as follows:

$$\eta(i, x) = d(i, j) \qquad \text{for } B_{j-1} \leq x < B_j, 1 \leq j \leq J. \tag{107}$$

If B is finite, the system is always stable. If B is infinite, the stability condition is:

$$\sum_{i=1}^{M} \pi_i d(i, J) < 0. \tag{108}$$

We assume this to hold if $B = \infty$ and study the steady distribution of the $\{(X(t), Z(t)), t \geq 0\}$ process. The results here are based on Elwalid and Mitra [9].

Let $F(x, i)$ and $F(x)$ be as defined by Equations (22) and (23). For $1 \leq j \leq J$, we use the notation

$$\begin{aligned}
F^j(x, i) &= F(x, i) \qquad \text{for } B_{j-1} < x < B_j, i \in S, & (109)\\
F^j(x) &= F(x) \qquad \text{for } B_{j-1} < x < B_j, & (110)\\
D^j &= diag(d(1, j), d(2, j), ..., d(M, j)), & (111)\\
S_+^j &= \{i \in S : d(i, j) > 0\}, & (112)\\
S_0^j &= \{i \in S : d(i, j) = 0\}, & (113)\\
S_-^j &= \{i \in S : d(i, j) < 0\}. & (114)
\end{aligned}$$

From the results of Section 3 we get the following theorem.

Theorem 13 *$\{F^j(x), 1 \leq j \leq J\}$ satisfy the following equations:*

$$\frac{dF^j(x)}{dx} D^j = F^j(x) Q, \tag{115}$$

with the following boundary conditions:

$$\begin{aligned}
F^1(0, i) &= 0 \qquad \text{if } i \in S_+^1, & (116)\\
F^j(B_j-, i) &= F^{j+1}(B_j+, i) \quad \text{if } i \in S_-^j \cap S_-^{j+1}, 1 \leq j \leq J-1 \text{ or} \\
&\qquad \text{if } i \in S_+^j \cap S_+^{j+1}, 1 \leq j \leq J-1. & (117)
\end{aligned}$$

If the buffer capacity is finite the additional boundary conditions are

$$F^J(B_J-, i) = \pi_i \qquad \text{if } i \in S_-^J. \tag{118}$$

Proof: See Elwalid and Mitra [9].

We follow the methodology of Section 3 to obtain the spectral representation for $F(x)$ given in the next theorem.

Theorem 14 *Let* $\{(\lambda_i^j, \phi_i^j), 1 \leq i \leq M, 1 \leq j \leq J\}$ *be the (eigenvalue, eigenvector) pairs for the following generalized eigenvalue problems:*

$$\phi(\lambda D^j - Q) = 0 \qquad 1 \leq j \leq J \ . \tag{119}$$

Then $F^j(x)$ *has the following spectral representation:*

$$F^j(x) = \sum_{i=1}^{M} a_i^j \phi_i^j e^{\lambda_i^j x} \qquad 1 \leq j \leq J \ , \tag{120}$$

where the scalars $\{a_i^j, 1 \leq i \leq M, 1 \leq j \leq J\}$ *are chosen to satisfy the boundary conditions in Equations (116) - (118). If the buffer capacity is infinite the conditions generated by Equation (118) are replaced by the following:*

$$a_i^J = 0 \qquad \text{if } Re(\lambda_i^J) > 0, \tag{121}$$

$$a_i^J = 1 \qquad \text{if } Re(\lambda_i^J) = 0. \tag{122}$$

Equation (117) says that the bivariate process $\{(X(t), Z(t)), t \geq 0\}$ has no mass at (B_j, i) if the drift (in state i) on both sides of B_j has the same sign. Otherwise there may be a positive mass at (B_j, i). This makes intuitive sense. Note that there are as many equations as there are unknown a_i^j's. Hence the above theorem gives a complete solution to the steady-state distribution of the bivariate process.

When $\eta(z, x)$ is not a step function in x, one can approximate it by a step function in x and use the above results. Hence we have an approximate numerical procedure for solving the general problem when the external environment is a CTMC.

Example 4. Consider an on-off source (see Example 1). Suppose it produces two types of fluid (at rates R^1 and R^2) when it is on. The type 2 fluid is always accepted in the buffer if there is space for it. The type 1 fluid is accepted only if the buffer content is less than a given threshold $0 < B_1 < B$. (Thus the type 2 fluid will suffer fewer losses than the type 1 fluid and hence will have a better QoS.) The buffer is emptied at a fixed rate c.

This situation fits into the model analyzed above with two-state CTMC as an external environment and a two-step drift function(i.e., $M = 2, J = 2$). The Q matrix is given in Example 1. The two drift matrices are given by

$$D^1 = \begin{bmatrix} R^1 + R^2 - c & 0 \\ 0 & -c \end{bmatrix}, \tag{123}$$

$$D^2 = \begin{bmatrix} R^2 - c & 0 \\ 0 & -c \end{bmatrix}. \tag{124}$$

For a solution (in the case of an infinite buffer) we refer the reader to Kulkarni et al [25].

As discussed in Section 4.3, one can develop the concepts of effective bandwidths for multiclass traffic using a shared buffer approach. The models developed in the current section have been found useful in area of multiplexing multipriority traffic. Some work in this direction is in Kulkarni et al [25, 24].

8 Further Work

8.1 Other Driving Processes

One possible extension is to consider a semi-Markov process as a driving process and extend the results of Section 3 to this case. However, the work of Chen and Yao [6] suggests that the analysis is going to be rather hard.

Another possibility is to extend the results of Section 6 to the case where the driving process is a bivariate process $\{(Z_1(t), Z_2(t)), t \geq 0\}$, with $Z_1(t)$ being a CTMC and $Z_2(t)$ an OU process. The drift function is the same as in Section 6. Such a driving process is motivated by the multiplexing problems where a large number of small sources (giving rise to the OU component) are multipliexed along with a small number of large sources (giving rise to the CTMC component) onto a single buffer. The solution promises to be extremely complicated.

8.2 State Dependent Drifts

The results of Section 7 can be extended to other driving processes. For example, the driving process can be the CTMC + White Noise as in Section 5 or it can be the OU process of Section 6. These models are motivated by the buffer sharing models as explained in Example 4.

Work is currently in progress on a process that satisfies the following stochastic differential equation:

$$dX(t) = d(Z(t), X(t)) + \sigma^2(Z(t), X(t))dB(t) \tag{125}$$

where $\{Z(t), t \geq 0\}$ is a CTMC. As in Section 7 we first concentrate on the case where $d(z, x)$ and $\sigma^2(z, x)$ are step functions of x.

8.3 Multiclass Fluid Models

This subsection is motivated by the desire to extend the fluid models to the multiclass case along the same line as in the multipriority queues. The simplest

case is to assume that there are K classes. When the external environment is in state z the fluid of class k arrives at rate $R(z,k)$. The buffer is emptied at a maximum rate of c. Let $X_k(t)$ be the amount of fluid of class k in the buffer at time t and define $X(t) = (X_1(t), X_2(t), ..., X_K(t))$. The aim is to study the limiting distribution of $\{(X(t), Z(t)), t \geq 0\}$. Of course we need to specify how the the K classes are treated. A simple case is the Full-Service-Static-Priority discipline, under which the highest priority fluid that is in the buffer is always served first at the maximum possible rate.

Zang [38] has attempted to solve this problem via transform techniques. The joint distribution of $X(t)$ is rather messy. Narayanan [31] has developed the transforms of the marginal steady-state distributions of $X_k(t)$. Note that $Y_k(t) = \sum_{r=1}^{k} X_r(t)$ is a standard fluid model driven by $Z(t)$, assuming class 1 is the highest priority and class K is the lowest priority class. Hence, the steady state expected values of $X_k(t)$ are readily available.

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