# First Order Galilean Superfluid Dynamics

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ABSTRACT: We study dynamics of (anomalous) Galilean superfluid up to first order in derivative expansion, both in parity-even and parity-odd sectors. We construct a relativistic system – null superfluid, which is a null fluid (introduced in [16]) with a spontaneously broken global U(1) symmetry. A null superfluid is in one to one correspondence with Galilean superfluid in one lower dimension, i.e. they have same symmetries, thermodynamics, constitutive relations and are related to each other by a mere choice of basis. The correspondence is based on null reduction, which is known to reduce the Poincaré symmetry of a theory to Galilean symmetry in one lower dimension. To perform this analysis, we use offshell formalism of (super)fluid dynamics, adopting it appropriately to null (super)fluids.

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# 1 | Introduction and Summary

Hydrodynamics is an effective description of low energy fluctuations of a quantum system around thermodynamic equilibrium. In this description, we assume the hydrodynamic system, known as a *fluid*, to be at a finite temperature, and study its fluctuations at length scales much larger than the mean free path of the system. In this limit and far away from any second order phase transition point, a fluid can be described by a small number of degrees of freedom known as hydrodynamic modes: temperature, chemical potential(s) and normalized fluid velocity. Various conserved currents of the system can then be written in terms of these hydrodynamic modes, arranged as a perturbative expansion in derivatives, known as *fluid constitutive relations*. At any particular order in derivative expansion, constitutive relations contain all the possible independent tensor structures allowed by symmetry at that order, multiplied with unknown coefficients known as transport coefficients. If the underlying quantum theory has a continuous global sym-

metry which is spontaneously broken in the ground state, then the low energy fluctuations can contain massless Goldstone modes corresponding to the broken symmetry. Therefore for fluids with a spontaneously broken symmetry, known as *superfluids*, hydrodynamic modes also contain these Goldstone modes. This leads to a considerable modification of the constitutive relations, adding new tensor structures containing the derivatives of the Goldstone modes and hence new transport coefficients. In this paper, we work out the most generic constitutive relations of a Galilean superfluid up to first order in the derivative expansion.

Superfluidity was first observed in liquid helium by [1, 2] in 1938, while studying its flow through a thin capillary. They observed that liquid helium flows through the capillary without any dissipation, hence inspiring the name "superfluid". Other than this dissipationless flow, superfluids have many more striking features, such as upon rotation they develop vortices (quasi-one-dimensional strings whose number is proportional to the externally imposed angular momentum). Furthermore, their specific heat shows a sudden change in behavior at a certain critical temperature. Above the critical temperature system behaves like an ordinary fluid, though as the temperature drops below the critical temperature, system undergoes a phase transition from the ordinary fluid phase to the superfluid phase.

Study of superfluid dynamics has been a topic of interest for a long time. First theory of superfluid dynamics was written down by London [?] in 1938, followed by a two-fluid model of superfluids proposed by Landau and Tisza [3, 4] in 1940s. They studied ideal superfluids in a non-relativistic setting, which was later generalized to describe a relativistic superfluid by [5–10]. The subject was recently revisited by [11–13], who re-derived the relativistic superfluid constitutive relations using the second law of thermodynamics and equilibrium partition functions. Among other interesting results, they found that up to first order in derivative expansion, a relativistic superfluid is characterized by pressure (at ideal order), 23 parity-even and 7 parity-odd first order transport coefficients and 2 undetermined constants including the anomaly constant (after imposing Onsager relations and CPT invariance these numbers drop down to 16 parity-even and 6 parity odd transport coefficients and one anomaly constant). See table (1) for a summary and §2 for more details.

In this paper, we perform a similar exercise for Galilean superfluids. We derive the constitutive relations for a Galilean superfluid consistent with the second law of thermodynamics, up to first order in derivative expansion, both in parity even and odd sectors. Study of Galilean superfluids is important because it provides a laboratory to probe many-body physics in extreme quantum regime with high-precision [14]. Relativistic effects are important in high-energy superfluids, where mass of the constituents is small compared to their kinetic energy, e.g. quark superfluidity in compact stars. In contrast, for low-energy systems such as liquid helium and ultra-cold atomic gases, a Galilean framework is more ideal.

Recently in [15, 16], we established that one can derive the most generic constitutive relations for an ordinary Galilean fluid starting from a relativistic system, namely a *null fluid* in one higher dimension, followed by a *null reduction*<sup>1</sup> [18]. Loosely speaking, null fluid is a fluid coupled to a background with fields: a metric  $g_{MN}$ , a U(1) gauge field  $A_M$  and a covariantly

<sup>&</sup>lt;sup>1</sup>Null reduction of an ordinary relativistic fluid gives us a constrained Galilean fluid as found in [17].

constant null isometry  $\mathcal{V} = \{V^M, \Lambda_V\}$  with  $V^M A_M + \Lambda_V = \text{constant}$ . We call this background a null background<sup>2</sup>. Theories on a null background, which we call null theories, are demanded to be invariant under  $\mathcal{V}$  preserving diffeomorphisms and gauge transformations. Upon performing null reduction, i.e. choosing a basis  $\{x^M\} = \{x^-, t, x^i\}$  such that  $\mathcal{V} = \{V = \partial_-, \Lambda_V = 0\}$ , these restricted transformations reduce to the well known Galilean transformations on the background spanned by coordinates  $\{t, x^i\}$ . It suggests that null theories are entirely equivalent to Galilean theories, and are related by merely this choice of basis. It follows that a fluid on null background – null fluid is entirely equivalent to a Galilean fluid. Their constitutive relations, conservation laws, thermodynamics etc. match exactly to all orders in derivative expansion. Due to presence of an additional vector field  $V^M$ , constitutive relations of a null fluid are vastly different from those of a relativistic fluid and contain many more transport coefficients. This accounts for the additional transport coefficients in a Galilean fluid as compared to a relativistic fluid, while at the same time establishing that the most generic Galilean fluid cannot be gained by null reduction of an ordinary relativistic fluid.

In this paper, we take the construction of null fluids one step further to include null superfluids, i.e. we construct a null fluid with a spontaneously broken U(1) symmetry. The corresponding Goldstone mode is a new field in the theory and modifies the constitutive relations of an ordinary null fluid. Once we have the constitutive relations for a null superfluid, corresponding Galilean superfluid constitutive relations follow trivially via null reduction. We find that up to first order in derivatives, a Galilean superfluid is described by pressure P (at ideal order), a total of 51 first order transport coefficients and two unknown constants including the anomaly constant. Out of these 51 coefficients, 38 lie in parity-even sector while 13 are in parity-odd sector. Furthermore, only 22 parity-even and 3 parity-odd coefficients are dissipative. Out of the nondissipative coefficients, 3 parity-even and 3 parity-odd coefficients describe equilibrium physics, while the remaining 13 parity-even and 7 parity-odd coefficients describe non-dissipative effects away from equilibrium. Finally, following the intuition from relativistic superfluids and known Galilean results in [20], there are hints that the 7 parity-even non-dissipative non-hydrostatic coefficients and 3 parity-odd dissipative coefficients are switched off using Onsager relations (imposing microscopic reversibility of field theories). This would imply that the parity-odd sector is purely non-dissipative. However, a detailed microscopic calculation is required to establish confidence in these Galilean Onsager relations, which we do not perform in this paper. In table (1), we have summarized the counting of transport coefficients for the most generic Galilean superfluid, along with a comparison with relativistic superfluids reviewed in §2, and known results for ordinary Galilean and relativistic fluids.

Another recent development in hydrodynamics is offshell formalism introduced by [21–23], which streamlines the analysis of constitutive relations in accordance with the second law of thermodynamics, up to arbitrarily high orders in derivative expansion. We have reviewed this formalism in §2. In a nutshell, for ordinary fluids the formalism requires us to consider a version of the

<sup>&</sup>lt;sup>2</sup>Here, definition of null backgrounds has been adapted to a torsionless spacetime. For backgrounds with torsion, look at [19].

	Relativistic	Relativistic	Galilean	Galilean
	Fluid	Superfluid	Fluid	Superfluid
Hydrostatic	$0 + \tilde{0}$	$2+\tilde{2}$	$0 + \tilde{0}$	$3+\tilde{3}$
Non-hydrostatic non-diss.	$0^* + \tilde{0}$	$7^*+ ilde{4}$	$1^* + \tilde{0}$	$13^* + \tilde{7}$
Dissipative	$2+\tilde{0}^*$	$14 + \tilde{1}^*$	$5+\tilde{0}^*$	$22 + \tilde{3}^*$
Total	$2 + \tilde{0} = 2$	$23 + \tilde{7} = 30$	$6 + \tilde{0} = 6$	$38 + \tilde{13} = 51$
Total (with Onsager)	$2 + \tilde{0} = 2$	$16 + \tilde{6} = 22$	$5 + \tilde{0} = 5$	$25 + \tilde{10} = 35$
Hydrostatic Constants	$\tilde{3} + \tilde{1}_{anomaly}$	$\tilde{1} + \tilde{1}_{anomaly}$	$\tilde{4} + \tilde{1}_{anomaly}$	$\tilde{1} + \tilde{1}_{anomaly}$

Table 1: Counting of the independent first order transport coefficients consistent with the second law of thermodynamics. The numbers with a "tilde" represent the parity-odd count (in 3 spatial dimensions) while the "un-tilde" numbers are the parity-even count. The coefficients with an "asterisk" drop out on imposing Onsager relations (microscopic time-reversal invariance). Finally, in the last row we have given the number of undetermined constants including the anomaly constant. In both relativistic and Galilean cases, we have gotten rid of a hydrostatic coefficient by redefinition of the U(1) phase  $\varphi$ .

second law of thermodynamics which is valid for thermodynamically non-isolated fluids,

$$\nabla_{\mu}J_{S}^{\mu} + \frac{u_{\mu}}{T} \left( \nabla_{\nu}T^{\mu\nu} - F^{\nu\rho}J_{\rho} - \mathcal{T}_{H}^{\mu\perp} \right) + \frac{\mu}{T} \left( \nabla_{\mu}J^{\mu} - \mathcal{J}_{H}^{\perp} \right) = \Delta \ge 0. \tag{1.1}$$

Since the fluid is not thermodynamically isolated, it is allowed to interact with its surrounding and hence conservation laws are no longer satisfied. Therefore the original second law  $\nabla_{\mu}J_{S}^{\mu} \geq 0$  has been modified with combinations of the conservation laws. We need to find the most generic constitutive relations for  $T^{\mu\nu}$ ,  $J^{\mu}$  allowed by symmetries (modulo terms related to each other by conservation laws) which satisfy eqn. (1.1) for some entropy current  $J_{S}^{\mu}$  and  $\Delta \geq 0$ . When we move to superfluids, we have an additional field  $\varphi$  (the Goldstone mode) which comes with its own equation of motion K=0, the Josephson equation. Going offshell in  $\varphi$ , conservation equations get modified by combinations of K, and the second law of thermodynamics for thermodynamically non-isolated superfluids takes the form (see [24] for more details),

$$\nabla_{\mu}J_{S}^{\mu} + \frac{u_{\mu}}{T} \left( \nabla_{\nu}T^{\mu\nu} - F^{\nu\rho}J_{\rho} - \mathcal{T}_{H}^{\mu\perp} - \xi^{\mu}K \right) + \frac{\mu}{T} \left( \nabla_{\mu}J^{\mu} - \mathcal{J}_{H}^{\perp} + K \right) = \Delta \ge 0. \tag{1.2}$$

Note that contrary to the philosophy of [21–23], though we have gone offshell in  $\varphi$  we have not modified the second law with a multiple of K. Rather, we require the second law of thermodynamics to be satisfied even for offshell configurations of  $\varphi$ . Next, we find the most generic "superfluid constitutive relations"  $T^{\mu\nu}$ ,  $J^{\mu}$ , K allowed by symmetries (modulo terms related to each other by conservation laws or the Josephson equation) which satisfy eqn. (1.2) for some entropy current  $J_S^{\mu}$  and  $\Delta \geq 0$ . In §3, we have extended this formalism to null (super)fluids, and used it to work out the constitutive relations of a null/Galilean superfluid up to first order in derivative expansion.

The paper is organized as follows: we start §2 with a review of offshell formalism for relativistic hydrodynamics. Readers well familiar with this formalism can skip to §2.2 where we have reviewed offshell formalism for relativistic superfluids and used it to work out respective constitutive relations up to first order in derivative expansion. Next in §3, we introduce offshell formalism for null superfluids and find respective constitutive relations up to first order in

derivative expansion. The null superfluid results have been reduced to Galilean superfluids in §4. These are the main results of this paper. Finally, we conclude with some discussion in §5. The paper contains two appendices: in appendix (A) we present equilibrium partition function for null superfluids and in appendix (B) we give details of some computations glossed over in the main text.

## 2 | Revisiting Relativistic Superfluids

Before starting with null superfluids, it is instructive to revisit the relativistic superfluids first. It will help us appreciate the similarities between the two systems, while at the same time allowing for an isolation of the differences. Needless to say, all the results in this section have already been worked out in the literature [11–13], however our approach will be slightly different. We will work in the "offshell formalism of hydrodynamics", which was introduced for ordinary (non-super) fluids in [21, 23], and later extended to superfluids in [24].

### 2.1 Offshell Formalism for Relativistic Ordinary Fluids

Let us begin with ordinary relativistic fluids. Consider a d-dimensional manifold  $\mathcal{M}_d$  equipped with the background fields: a metric  $g_{\mu\nu}$  and a U(1) gauge field  $A_{\mu}$ . Physical theories coupled to  $\mathcal{M}_d$  are required to be invariant under diffeomorphisms and U(1) gauge transformations. These act on the said background fields as,

$$\delta_{\mathcal{X}}g_{\mu\nu} = \pounds_{\mathcal{X}}g_{\mu\nu} = \nabla_{\mu}\chi_{\nu} + \nabla_{\nu}\chi_{\mu}, \qquad \delta_{\mathcal{X}}A_{\mu} = \pounds_{\mathcal{X}}A_{\mu} + \partial_{\mu}\Lambda_{\mathcal{X}} = \partial_{\mu}(\Lambda_{\mathcal{X}} + \chi^{\nu}A_{\nu}) + \chi^{\nu}F_{\nu\mu}, \quad (2.1)$$

for some diffeomorphism and U(1) gauge parameters  $\mathcal{X} = \{\chi^{\mu}, \Lambda_{\chi}\}$  respectively. In this work we will only be interested in a particular class of these theories – fluids, which are are the universal near equilibrium limit of quantum field theories. Near equilibrium, the spectrum of any quantum field theory on  $\mathcal{M}_d$  must contain an energy momentum tensor  $T^{\mu\nu}$  and a charge current  $J^{\mu}$ . These quantities satisfy a set of conservation laws (here  $\nabla_{\mu}$  is the covariant derivative associated with  $g_{\mu\nu}$ ,  $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$  is the field strength associated with  $A_{\mu}$  and  $T_{H}^{\mu\perp}$ ,  $J_{H}^{\perp}$  are Hall currents carrying the anomalous contribution to the conservation equations),

$$\nabla_{\mu} T^{\mu\nu} - F^{\nu\rho} J_{\rho} - T_{H}^{\nu\perp} = 0, \qquad \nabla_{\mu} J^{\mu} - J_{H}^{\perp} = 0,$$
 (2.2)

provided that the system is thermodynamically isolated. In fact, eqn. (2.2) can be taken as a definition of thermodynamic isolation for near equilibrium quantum systems. The conservation laws eqn. (2.2) can also be thought of as a 'near equilibrium version' of the first law of thermodynamics, which imposes the conservation of not just energy, but also momentum and U(1) charge. Formally, we define an (ordinary) fluid as a near equilibrium system characterized by the currents  $T^{\mu\nu}$ ,  $J^{\mu}$ , with dynamics given by the conservation laws eqn. (2.2) imposed as the 'equations of motion'. Since eqn. (2.2) are (d+1) equations in d dimensions, they can provide dynamics for a fluid described by an arbitrary set of (d+1) variables. We choose these to be a normalized fluid velocity  $u^{\mu}$  (with  $u^{\mu}u_{\mu} = -1$ ), a temperature T and a chemical potential

 $\mu$ , collectively known as the hydrodynamic fields (modes). A fluid hence is completely characterized by a gauge-invariant expression of  $T^{\mu\nu}$ ,  $J^{\mu}$  in terms of  $g_{\mu\nu}$ ,  $A_{\mu}$ ,  $u^{\mu}$ , T,  $\mu$ , known as the hydrodynamic constitutive relations. The near equilibrium assumption allows us to arrange these constitutive relations as a perturbative expansion in derivatives (known as derivative or gradiant expansion), consistently truncated at a finite order in derivatives.

Being a thermodynamic system, a fluid is also required to satisfy a version of the second law of thermodynamics, stating that there must exist an entropy current  $J_S^{\mu}$  whose divergence is positive semi-definite everywhere, i.e.,

$$\nabla_{\mu} J_S^{\mu} = \Delta \ge 0, \tag{2.3}$$

as long as the fluid is thermodynamically isolated (i.e. conservation laws eqn. (2.2) or equivalently the first law(s) of thermodynamics are satisfied). The job of hydrodynamics now is to find the most general constitutive relations  $T^{\mu\nu}$ ,  $J^{\mu}$  and an associated  $J_S^{\mu}$ ,  $\Delta$  order by order in derivative expansion, such that eqn. (2.3) is satisfied for thermodynamically isolated fluids. This task has been extensively undertaken in the literature [25–30].

The problem stated in this language however, turns out to be increasingly hard to solve as we go to 2nd or higher orders in derivative expansion [31]. Fortunately, it was realized in [21] that most of the complication in the aforementioned computation comes from the fact that we need to maintain the thermodynamic isolation (i.e. satisfy the conservation equations) perturbatively at every order. A much easier problem to solve is to allow for the fluid to interact with its surroundings, i.e. break the conservation laws eqn. (2.2) by introducing an arbitrary external momentum  $P_{ext}^{\mu}$  and a charge  $Q_{ext}$  source,

$$\nabla_{\mu} T^{\mu\nu} - F^{\nu\rho} J_{\rho} - T_{H}^{\nu\perp} = P_{ext}^{\nu}, \qquad \nabla_{\mu} J^{\mu} - J_{H}^{\perp} = Q_{ext}.$$
 (2.4)

The LHS of the second law in eqn. (2.3) will also need to be augmented with an arbitrary combination of  $P_{ext}^{\mu}$ ,  $Q_{ext}$  for the inequality to be satisfied,

$$\nabla_{\mu}J_{S}^{\mu} + \beta_{\nu}P_{ext}^{\nu} + (\Lambda_{\beta} + A_{\mu}\beta^{\mu})Q_{ext} = \Delta \ge 0,$$

$$\implies \nabla_{\mu}J_{S}^{\mu} + \beta_{\nu}\left(\nabla_{\mu}T^{\mu\nu} - F^{\nu\rho}J_{\rho} - T_{H}^{\mu\perp}\right) + (\Lambda_{\beta} + A_{\mu}\beta^{\mu})\left(\nabla_{\mu}J^{\mu} - J_{H}^{\perp}\right) = \Delta \ge 0, \quad (2.5)$$

for some fields  $\mathcal{B} = \{\beta^{\mu}, \Lambda_{\beta}\}$ . This version of the second law is known as the *offshell second law* of thermodynamics, because the conservation laws, which are imposed as equations of motion on the fluid, are not required to be satisfied. Eqn. (2.5) can be recasted into a yet another useful form by defining a free energy current  $G^{\mu}$  as,

$$-\frac{G^{\mu}}{T} = N^{\mu} = J_S^{\mu} + \beta_{\nu} T^{\mu\nu} + (\Lambda_{\beta} + A_{\nu}\beta^{\nu}) J^{\mu}, \quad -\frac{G_H^{\perp}}{T} = N_H^{\perp} = \beta_{\mu} T_H^{\mu\perp} + (\Lambda_{\beta} + A_{\nu}\beta^{\nu}) J_H^{\perp}.$$
(2.6)

Having done that, eqn. (2.5) implies a free energy conservation,

$$\nabla_{\mu} N^{\mu} - \mathcal{N}_{\mathcal{H}}^{\perp} = \frac{1}{2} T^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} + J^{\mu} \delta_{\mathcal{B}} A_{\mu} + \Delta, \qquad \Delta \ge 0, \tag{2.7}$$

where similar to eqn. (2.1) we have defined,

$$\delta_{\mathcal{B}}g_{\mu\nu} = \pounds_{\beta}g_{\mu\nu} = \nabla_{\mu}\beta_{\nu} + \nabla_{\nu}\beta_{\mu}, \qquad \delta_{\mathcal{B}}A_{\mu} = \pounds_{\beta}A_{\mu} + \partial_{\mu}\Lambda_{\beta} = \partial_{\mu}\left(\Lambda_{\beta} + \beta^{\nu}A_{\nu}\right) + \beta^{\nu}F_{\nu\mu}. \tag{2.8}$$

Recall that the hydrodynamic fields  $u^{\mu}$ , T,  $\mu$  introduced earlier were some arbitrary (d+1) fields chosen to describe the fluid. Like in any field theory, they are permitted to admit an arbitrary redefinition among themselves without changing the physics. This huge amount of freedom can be fixed by explicitly choosing,

$$T = \frac{1}{\sqrt{-\beta^{\nu}\beta_{\nu}}}, \qquad u^{\mu} = \frac{\beta^{\mu}}{\sqrt{-\beta^{\nu}\beta_{\nu}}}, \qquad \mu = \frac{\Lambda_{\beta} + A_{\mu}\beta^{\mu}}{\sqrt{-\beta^{\nu}\beta_{\nu}}}, \tag{2.9}$$

or conversely,

$$\beta^{\mu} = \frac{1}{T}u^{\mu}, \qquad \Lambda_{\beta} = \frac{1}{T}\mu - A_{\mu}\beta^{\mu}. \tag{2.10}$$

As a consequence of this choice,  $\mathcal{B} = \{\beta^{\mu}, \Lambda_{\beta}\}$  is just a renaming of the hydrodynamic fields. Finally, we can find the most general gauge-invariant expression of the currents  $T^{\mu\nu}$ ,  $J^{\mu}$  in terms of  $g_{\mu\nu}$ ,  $A_{\mu}$ ,  $\beta^{\mu}$ ,  $\Lambda_{\beta}$  arranged in a derivative expansion, along with an associated  $N^{\mu}$ ,  $\Delta$  such that eqn. (2.7) is satisfied. There however is a caveat in this way of thinking: these  $T^{\mu\nu}$ ,  $J^{\mu}$  are not just the constitutive relations of a fluid; they also contain information about the external sources  $P^{\mu}_{ext}$ ,  $Q_{ext}$ . One way to circumvent this problem is to pick a set of terms which might potentially appear in  $T^{\mu\nu}$ ,  $J^{\mu}$  and can be eliminated using the conservation laws, and only consider the solutions  $T^{\mu\nu}$ ,  $J^{\mu}$  of eqn. (2.7) (for some  $N^{\mu}$ ,  $\Delta$ ) which do not involve these terms or their derivatives.  $T^{\mu\nu}$ ,  $J^{\mu}$  thus obtained are guaranteed to be the constitutive relations of a fluid, as they will be free from any  $P^{\mu}_{ext}$ ,  $Q_{ext}$  dependence.

Authors in [22, 23] illustrated a consistent mechanism to find the most generic constitutive relations of a fluid up to arbitrarily high orders in derivative expansion, which satisfies eqn. (2.7). They further classified these constitutive relations in eight exhaustive classes, which we will not have scope to review here. Instead, in the following subsection we will review the offshell analysis of relativistic superfluids which has been introduced in [24], and later adapt it to Galilean superfluids.

### 2.2 Offshell Formalism for Relativistic Superfluids

Let us now review some essential aspects of the offshell formalism for a relativistic superfluid following the work of [24], and use it to re-derive the respective constitutive relations up to first order in derivative expansion [11–13]. As we have already mentioned in the introduction, a superfluid is a phase of the fluid where the global U(1) symmetry of the microscopic theory gets spontaneously broken in the ground state due to condensation of a charged scalar operator. The U(1) phase  $\varphi$  of the scalar operator becomes a new field in the theory, along with  $u^{\mu}$ , T,  $\mu$  on which the respective constitutive relations can depend. Under an infinitesimal gauge transformation and diffeomorphism,  $\varphi$  transforms as  $\delta_{\chi}\varphi = \chi^{\mu}\partial_{\mu}\varphi - \Lambda_{\chi}$ , with covariant derivative,

$$\xi_{\mu} = \partial_{\mu}\varphi + A_{\mu},\tag{2.11}$$

commonly known as the *superfluid velocity*. Just like the dynamics of  $u^{\mu}$ , T,  $\mu$  is given by the conservation equations eqn. (2.2),  $\varphi$  comes with its own equation of motion <sup>3</sup>,

$$K = 0, (2.12)$$

 $<sup>{}^3</sup>K=0$  should be thought of as a placeholder for the Josephson junction condition  $u^{\mu}\xi_{\mu}=\mu+\mathcal{O}(\partial)$ , which provides dynamics for the U(1) phase  $\varphi$  in the conventional treatment of superfluids. At the moment however,

We will be particularly interested in the 'offshell' configurations of the field  $\varphi$ , which we define as the superfluid configurations for which  $K \neq 0$ . As was suggested by [24], conservation laws for these configurations modify to,

$$\nabla_{\mu} T^{\mu\nu} = F^{\nu\rho} J_{\rho} + T_{H}^{\nu\perp} + \xi^{\nu} K, \qquad \nabla_{\mu} J^{\mu} = J_{H}^{\perp} - K,$$
 (2.13)

which trivially turn back to their original form in eqn. (2.2) when K=0. The claim is that even the  $\varphi$ -offshell configurations of a superfluid satisfy the second law of thermodynamics, i.e. there exists an entropy current  $J_S^{\mu}$  whose divergence is positive semi-definite, i.e.,

$$\nabla_{\mu} J_S^{\mu} = \Delta \ge 0, \tag{2.14}$$

as long as the superfluid is thermodynamically isolated (i.e. eqn. (2.13) are satisfied), irrespective of K being zero. Rest of the analysis follows exactly like ordinary fluids; on allowing the superfluid to interact with its surroundings, the second law modifies to,

$$\nabla_{\mu} J_{S}^{\mu} + \beta_{\nu} \left( \nabla_{\mu} T^{\mu\nu} - F^{\nu\rho} J_{\rho} - T_{H}^{\mu\perp} - \xi^{\nu} K \right) + (\Lambda_{\beta} + A_{\sigma} \beta^{\sigma}) \left( \nabla_{\mu} J^{\mu} - J_{H}^{\perp} + K \right) = \Delta \ge 0. \quad (2.15)$$

In terms of free energy current however, we get,

$$\nabla_{\mu}N^{\mu} - N_{H}^{\perp} = \frac{1}{2}T^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + J^{\mu}\delta_{\mathcal{B}}A_{\mu} + K\delta_{\mathcal{B}}\varphi + \Delta, \qquad \Delta \ge 0, \tag{2.16}$$

where,

$$\delta_{\mathcal{B}}\varphi = \beta^{\mu}\partial_{\mu}\varphi - \Lambda_{\beta} = \frac{1}{T}\left(u^{\mu}\xi_{\mu} - \mu\right). \tag{2.17}$$

Similar to the ordinary fluid, we should now consider the most generic expressions for  $T^{\mu\nu}$ ,  $J^{\mu}$ , K in terms of  $g_{\mu\nu}$ ,  $A_{\mu}$ ,  $\beta^{\mu}$ ,  $\Lambda_{\beta}$ ,  $\varphi$  arranged in a derivative expansion, along with an associated  $N^{\mu}$ ,  $\Delta$  such that eqn. (2.16) is satisfied. However, these  $T^{\mu\nu}$ ,  $J^{\mu}$ , K will not be the constitutive relations of a superfluid, as they will also have information about the surroundings. The true constitutive relations will be gained by considering those solutions to eqn. (2.16) which do not involve a chosen set of terms that can be eliminated using the conservation equations eqn. (2.13) and the  $\varphi$  equation of motion eqn. (2.12).

We will now embark on the quest of finding these constitutive relations up to first order in the derivative expansion. [24] provides a complete classification and construction of the superfluid constitutive relations satisfying eqn. (2.16) up to arbitrarily high orders in derivative expansion. In this work however, we are only concerned with the (Galilean) superfluids up to first derivative order, which can be analyzed directly by brute force without involving the technicalities of [24].

### 2.2.1 Josephson Equation

In the study of superfluids, the U(1) phase  $\varphi$  is generally taken to be order -1 in the derivative expansion, while its covariant derivative  $\xi_{\mu}$  is taken to be order 0. The reason being that the true dynamical degrees of freedom are encoded in the fluctuations of  $\varphi$  along the U(1) circle,

we will allow for an arbitrary K treating it as yet another 'current' besides  $T^{\mu\nu}$ ,  $J^{\mu}$  in the theory, and will later establish that the second law of thermodynamics forces K to take the Josephson form.

and not in  $\varphi$  itself. It implies that the  $K\delta_{\mathcal{B}}\varphi$  term in the free energy conservation eqn. (2.16) is allowed to be order zero, if K has an order 0 term. This gives us the unique solution to eqn. (2.16) at zero derivative order,

$$N^{\mu}, T^{\mu\nu}, J^{\mu} = \mathcal{O}(\partial^{0}), \qquad K = -\alpha \delta_{\mathcal{B}} \varphi + \mathcal{O}(\partial), \qquad \Delta = \alpha (\delta_{\mathcal{B}} \varphi)^{2} + \mathcal{O}(\partial),$$
 (2.18)

for some "transport coefficient"  $\alpha \geq 0$ . Note that the  $\varphi$  equation of motion at this order will read  $K = -\alpha \delta_{\mathcal{B}} \varphi + \mathcal{O}(\partial) = 0$ , implying,

$$\delta_{\mathcal{B}}\varphi = \frac{1}{T}\left(u^{\mu}\xi_{\mu} - \mu\right) = \mathcal{O}(\partial) \implies u^{\mu}\xi_{\mu} = \mu + \mathcal{O}(\partial).$$
 (2.19)

This is the well known Josephson equation. This condition also ensures that  $\Delta$  is at least  $\mathcal{O}(\partial)$ , avoiding "ideal superfluid dissipation". From this point onward, it would be beneficial to think of  $\delta_{\mathcal{B}}\varphi$  as an order 1 data in derivative expansion rather than 0.

### 2.3 Ideal Relativistic Superfluids

Let us now move on to the ideal superfluids, i.e. superfluid constitutive relations that satisfy the free energy conservation eqn. (2.16) at first derivative order. At ideal order, the most generic tensorial form of various quantities appearing in eqn. (2.16) can be written as,

$$T^{\mu\nu} = (E+P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + R_s\xi^{\mu}\xi^{\nu} + \lambda(u^{\mu}\xi^{\nu} + u^{\nu}\xi^{\mu}) + \mathcal{O}(\partial),$$

$$J^{\mu} = Qu^{\mu} + Q_s\xi^{\mu} + \mathcal{O}(\partial),$$

$$K = -\alpha\delta_{\mathcal{B}}\varphi + K_{ideal} + \mathcal{O}(\partial),$$

$$N^{\mu} = Nu^{\mu} + N_s\xi^{\mu} + \mathcal{O}(\partial),$$

$$\Delta = (\alpha\delta_{\mathcal{B}}\varphi)^2 + \Delta_{ideal} + \mathcal{O}(\partial^2),$$
(2.20)

where E, P,  $R_s$ ,  $\lambda$ , Q,  $Q_s$ ,  $K_{ideal}$ , N,  $N_s$  are functions of T,  $\mu$  and  $\mu_s \equiv -\frac{1}{2}\xi^{\mu}\xi_{\mu}$ . We have omitted the only other possible scalar  $\delta_B\varphi$  in the functional dependence, because using the  $\varphi$  equation of motion we know that it is no longer an independent quantity. Plugging eqn. (2.20) in eqn. (2.16) we can find,

$$(Q_{s} + R_{s})\xi^{\mu} \left(\nabla_{\mu}\nu + \frac{1}{T}u^{\nu}F_{\nu\mu}\right) + \lambda\xi^{\mu} \left(\frac{1}{T^{2}}\nabla_{\mu}T + u^{\nu}\nabla_{\nu}\left(\frac{u_{\mu}}{T}\right)\right)$$

$$+ \nabla_{\mu} \left(\left(\frac{P}{T} - N\right)u^{\mu}\right) + \frac{1}{T}u^{\mu} \left(\nabla_{\mu}E - T\nabla_{\mu}S - \mu\nabla_{\mu}Q + R_{s}\nabla_{\mu}\mu_{s}\right)$$

$$+ \nabla_{\mu} \left(\delta_{\mathcal{B}}\varphi R_{s}\xi^{\mu} - N_{s}\xi^{\mu}\right) + \left(K_{ideal} - \nabla_{\mu}(R_{s}\xi^{\mu})\right)\delta_{\mathcal{B}}\varphi + \Delta_{ideal} = 0, \quad (2.21)$$

where we have defined S through the "Euler equation",

$$E + P = ST + Q\mu. \tag{2.22}$$

Eqn. (2.21) will imply a set of relations among various coefficients,

$$Q_s = -R_s, \quad \lambda = 0, \quad N = \frac{P}{T}, \quad N_s = \delta_{\mathcal{B}} \varphi R_s, \quad K_{ideal} = \nabla_{\mu} (R_s \xi^{\mu}), \quad \Delta_{ideal} = 0, \quad (2.23)$$

and the "first law of thermodynamics",

$$dE = TdS + \mu dQ - R_s d\mu_s, \qquad (2.24)$$

giving physical meaning to the quantities we have introduced in eqn. (2.20). Finally, we have the full set of superfluid constitutive relations up to ideal order satisfying the second law,

$$T^{\mu\nu} = (E+P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + R_s\xi^{\mu}\xi^{\nu} + \mathcal{O}(\partial),$$

$$J^{\mu} = Qu^{\mu} - R_s\xi^{\mu} + \mathcal{O}(\partial),$$

$$K = -\alpha\delta_{\mathcal{B}}\varphi + \nabla_{\mu}(R_s\xi^{\mu}) + \mathcal{O}(\partial),$$

$$N^{\mu} = \frac{P}{T}u^{\mu} + \delta_{\mathcal{B}}\varphi R_s\xi^{\mu} + \mathcal{O}(\partial),$$

$$J^{\mu}_{S} = N^{\mu} - \frac{1}{T}(T^{\mu\nu}u_{\nu} + \mu J^{\mu}) = Su^{\mu} + \mathcal{O}(\partial),$$

$$\Delta = \mathcal{O}(\partial^2).$$
(2.25)

These are the well known ideal superfluid constitutive relations. Note that we have included first order terms in K,  $N^{\mu}$  which can be ignored when talking about the ideal order, but are required for internal consistency with eqn. (2.16). The  $\varphi$  equation of motion K = 0 will imply,

$$\alpha \delta_{\mathcal{B}} \varphi = \nabla_{\mu} (R_s \xi^{\mu}) + \mathcal{O}(\partial) \quad \Longrightarrow \quad u^{\mu} \xi_{\mu} = \mu + \frac{T}{\alpha} \nabla_{\mu} (R_s \xi^{\mu}) + \mathcal{O}(\partial), \tag{2.26}$$

which is a first order correction to the Josephson equation. Note however that this equation can admit further one derivative corrections due to the first order constitutive relations discussed in the next subsection; the correction mentioned here is only how the ideal superfluid transport affects the Josephson equation. The conservation laws on the other hand are complete up to the first order in derivatives,

$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}(E+P)T^{2}\beta_{\mu}\right) + QT\delta_{\mathcal{B}}A_{\mu} = -\xi_{\nu}\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^{2}),$$

$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}QT\right) = \alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^{2}).$$
(2.27)

These equations provide a set of relations between  $\delta_{\mathcal{B}}\varphi$ ,  $\delta_{\mathcal{B}}g_{\mu\nu}$  and  $\delta_{\mathcal{B}}A_{\mu}$ , which can be used to eliminate a vector  $u^{\mu}\delta_{\mathcal{B}}g_{\mu\nu}$  and a scalar  $u^{\mu}\delta_{\mathcal{B}}A_{\mu}$  (see table (2)) from the first order constitutive relations. On the other hand, we choose to eliminate the scalar data  $\nabla_{\mu}(R_s\xi^{\mu})$  using the  $\varphi$  equation of motion.

### 2.4 First Derivative Corrections to Relativistic Superfluids

Moving on to the one derivative superfluids, let us schematically represent various quantities appearing in eqn. (2.16) up to the first order in derivatives as,

$$T^{\mu\nu} = \left[ (E+P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + R_{s}\xi^{\mu}\xi^{\nu} \right] + \mathcal{T}^{\mu\nu} + \mathcal{O}(\partial^{2}),$$

$$J^{\mu} = \left[ Qu^{\mu} - R_{s}\xi^{\mu} \right] + \mathcal{J}^{\mu} + \mathcal{O}(\partial^{2}),$$

$$K = \left[ -\alpha\delta_{\mathbb{B}}\varphi + \nabla_{\mu}(R_{s}\xi^{\mu}) \right] + \mathcal{K} + \mathcal{O}(\partial^{2}),$$

$$N^{\mu} = \left[ \frac{P}{T}u^{\mu} + \delta_{\mathbb{B}}\varphi R_{s}\xi^{\mu} \right] + \mathcal{N}^{\mu} + \mathcal{O}(\partial^{2}),$$

$$\Delta = \alpha(\delta_{\mathbb{B}}\varphi)^{2} + \mathcal{D},$$
(2.28)

V	Vanishing at Equilibrium – Onshell Independent					
$S_1$	$rac{T}{2} ilde{P}^{\mu u}\delta_{\mathbb{B}}g_{\mu u}$	$ ilde{P}^{\mu u} abla_{\mu}u_{ u}$				
$S_2$	$\frac{T}{2}\zeta^{\mu}\zeta^{ u}\delta_{\mathbb{B}}g_{\mu u}$	$\zeta^{\mu}\zeta^{ u} abla_{\mu}u_{ u}$				
$S_3$	$T\zeta^{\mu}\delta_{\mathbb{B}}A_{\mu}$	$\zeta^{\mu} \left( T \nabla_{\mu} \nu + u^{\nu} F_{\nu \mu} \right)$				
$S_4$	$T\delta_{\mathcal{B}}arphi$					
$V_1^{\mu}$	$T\tilde{P}^{\mu\nu}\zeta^{\rho}\delta_{\mathbb{B}}g_{\nu\rho}$					
$V_2^{\mu}$	$T\tilde{P}^{\mu u}\delta_{\mathbb{B}}A_{\mu}$	$T\tilde{P}^{\mu\nu}\delta_{\mathcal{B}}A_{\mu}$ $\tilde{P}^{\mu\nu}\left(T\nabla_{\nu}\nu+u^{\sigma}F_{\sigma\nu}\right)$				
$\sigma^{\mu  u}$	$\frac{T}{2}\tilde{P}^{ ho\langle\mu}\tilde{P}^{ u angle\sigma}\delta_{\mathbb{B}}g_{ ho\sigma}$	$\tilde{P}^{\mu\rho}\tilde{P}^{\nu\sigma}\left(\nabla_{(\rho}u_{\sigma)} - \frac{1}{d-2}\tilde{P}_{\rho\sigma}S_1\right)$				
$\tilde{V}_1^{\mu}$		$\epsilon^{\mu u ho\sigma}u_{ u}\zeta_{ ho}V_{1,\sigma}$				
$ ilde{V}_2^\mu$		$\epsilon^{\mu u ho\sigma}u_ u\zeta_ ho V_{2,\sigma}$				
$\tilde{\sigma}^{\mu\nu}$	$\epsilon$	$\epsilon^{\mu\nu\rho\sigma} u_{\nu} \zeta_{\rho} V_{2,\sigma}$ $\epsilon^{(\mu \rho\sigma\tau} u_{\rho} \zeta_{\sigma} \sigma_{\tau}^{\nu)}$				
7	Vanishing at Equilib	orium – Onshell Dependent				
$S_5$	$\frac{T}{2}u^{\mu}u^{\nu}\delta_{\mathcal{B}}g_{\mu\nu}$	$\frac{1}{T}u^{\mu}\nabla_{\mu}T$				
$S_6$	$Tu^{\mu}\delta_{\mathcal{B}}A_{\mu}$	$Tu^{\mu}\nabla_{\mu}\nu$				
$S_7$	$T\zeta^{\mu}u^{\nu}\delta_{\mathcal{B}}g_{\mu\nu}$	$\zeta^{\nu} \left( \frac{1}{T} \nabla_{\nu} T + u^{\sigma} \nabla_{\sigma} u_{\nu} \right)$				
$V_3^\mu = \tilde{V}_3^\mu$	$T\tilde{P}^{\mu\nu}u^{\rho}\delta_{\mathfrak{B}}g_{\nu\rho}$	$\frac{\zeta^{\nu} \left(\frac{1}{T} \nabla_{\nu} T + u^{\sigma} \nabla_{\sigma} u_{\nu}\right)}{\tilde{P}^{\mu\nu} \left(\frac{1}{T} \nabla_{\nu} T + u^{\sigma} \nabla_{\sigma} u_{\nu}\right)}$				
$ ilde{V}_3^\mu$		$\epsilon^{\mu u ho\sigma}u_{ u}\zeta_{ ho}V_{3,\sigma}$				
	Surviving	g at Equilibrium				
$S_{e,1}$		$\frac{1}{T}\zeta^{\mu}\partial_{\mu}T$				
$S_{e,2}$		$T\zeta^{\mu}\partial_{\mu} u$				
$S_{e,3}$		$\zeta^{\mu}\partial_{\mu}\hat{\mu}_{s}$				
$S_{e,4}$		$ abla_{\mu}\zeta^{\mu}$				
÷		:				
$V_{e,1}^{\mu}$		$\frac{1}{T}\tilde{P}^{\mu\nu}\partial_{\nu}T$				
$V_{e,2}^{\mu}$		$T ilde{P}^{\mu u}\partial_{ u} u$				
:	:					
$\tilde{S}_{e,1}$	$T\epsilon^{\mu u ho\sigma}\zeta_{\mu}u_{ u}\partial_{ ho}u_{\sigma}$					
$\tilde{S}_{e,2}$	$rac{1}{2}T\epsilon^{\mu u ho\sigma}\zeta_{\mu}u_{ u}F_{ ho\sigma}$					
:	:					
$\tilde{V}_{e,1}^{\mu}$	$T\tilde{P}^{\mu}_{\  au}\epsilon^{ au u ho\sigma}u_{ u}\partial_{ ho}u_{\sigma}$					
$ ilde{V}_{e,2}^{\mu}$	$rac{1}{2}T ilde{P}_{ au}^{\mu}\epsilon^{ au u ho\sigma}u_{ u}F_{ ho\sigma}$					
$\tilde{V}_{e,3}^{\mu}$	$T\tilde{P}^{\mu}_{\  au}\epsilon^{ au u ho\sigma}\xi_{ u}\partial_{ ho}u_{\sigma}$					
$ ilde{V}_{e,4}^{\mu}$	$rac{1}{2}T ilde{P}_{ au}^{\mu}\epsilon^{ au u ho\sigma}\xi_{ u}F_{ ho\sigma}$					
:	:					
	<u> </u>					

**Table 2:** Independent first order data for relativistic superfluids. We have not enlisted, neither would we need, all the independent data surviving at equilibrium.

where the corrections  $\mathcal{T}^{\mu\nu}$ ,  $\mathcal{J}^{\mu}$ ,  $\mathcal{K}$ ,  $\mathcal{N}^{\mu}$ ,  $\mathcal{D}$  have exactly one derivative in every term. Plugging these in the eqn. (2.16) we can get an equation among the corrections,

$$\nabla_{\mu} \mathcal{N}^{\mu} - \mathcal{N}_{\mathcal{H}}^{\perp} = \frac{1}{2} \mathcal{T}^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} + \mathcal{J}^{\mu} \delta_{\mathcal{B}} A_{\mu} + \mathcal{K} \delta_{\mathcal{B}} \varphi + \mathcal{D} + \mathcal{O}(\partial^{3}). \tag{2.29}$$

We will now attempt to find all the solutions to this equation, hence recovering the superfluid constitutive relations up to the first order in derivatives.

### 2.4.1 Parity Even

We can find the most general parity even solution of eqn. (2.29) in 2 steps (note that  $N_H^{\perp}$  is parity odd): (1) first we write down the most general allowed parity-even  $\mathcal{N}^{\mu}$  and find a set of constitutive relations pertaining to that, and (2) then find the most general parity-even constitutive relations which satisfy eqn. (2.29) with  $\mathcal{N}^{\mu} = 0$ .

1. One can check that the most general form of  $\mathcal{N}^{\mu}$  (whose divergence only contains product of derivatives and has at least one  $\delta_{\mathcal{B}}$  per term) can be written as,

$$\mathcal{N}^{\mu} = 2f_1 u^{[\mu} \xi^{\nu]} \frac{1}{T^2} \partial_{\nu} T + 2f_2 u^{[\mu} \xi^{\nu]} \partial_{\nu} \left(\frac{\mu}{T}\right) + 2f_3 u^{[\mu} \xi^{\nu]} \partial_{\nu} R_s + \nabla_{\nu} \left(f_4 u^{[\mu} \xi^{\nu]}\right), \quad (2.30)$$

where f's are functions of T,  $\nu = \mu/T$  and  $\hat{\mu}_s = -\frac{1}{2}\zeta^{\mu}\zeta_{\mu}$  with  $\zeta^{\mu} = P^{\mu\nu}\xi_{\nu}$  ( $P^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}$  is the projection operator away from the fluid velocity). Note that,

$$\hat{\mu}_s = -\frac{1}{2} \zeta^{\mu} \zeta_{\mu} = -\frac{1}{2} \xi^{\mu} \xi_{\mu} - \frac{1}{2} (\xi^{\mu} u_{\mu})^2 = \mu_s - \frac{1}{2} (\mu + T \delta_{\mathcal{B}} \varphi)^2.$$
 (2.31)

Out of the four terms in eqn. (2.30), the last one has trivially zero divergence and hence can be ignored. The third term on the other hand can be removed by elimination of  $\nabla_{\mu}(R_s\xi^{\mu})$  using the  $\varphi$  equation of motion. Computing the divergence of the remaining terms in  $\mathcal{N}^{\mu}$  and comparing them to eqn. (2.29), we can directly read out the corresponding superfluid constitutive relations (the symbol ' $\ni$ ' represents that they are not yet the complete solutions of eqn. (2.29); we still have to add the terms with  $\mathcal{N}^{\mu} = 0$ ),

$$\mathcal{T}^{\mu\nu} \ni u^{\mu} u^{\nu} \left( \sum_{i=1}^{2} \alpha_{E,i} S_{e,i} - \frac{1}{T} \nabla_{\sigma} (T f_{1} \zeta^{\sigma}) \right) + \left( \zeta^{\mu} \zeta^{\nu} - 2 (u^{\rho} \xi_{\rho}) u^{(\mu} \zeta^{\nu)} \right) \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i}$$

$$+ \tilde{P}^{\mu\nu} \sum_{i=1}^{2} f_{i} S_{e,i} - 2 \xi^{(\mu} \sum_{i=1}^{2} f_{i} V_{e,1}^{\nu)} + 2 u^{(\mu} \zeta^{\nu)} \sum_{i=1}^{2} f_{i} S_{4+i},$$

$$\mathcal{J}^{\mu} \ni u^{\mu} \left( \sum_{i=1}^{2} \alpha_{Q,i} S_{e,i} - \frac{1}{T} \nabla_{\nu} (T f_{2} \zeta^{\nu}) \right) - \zeta^{\mu} \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} f_{i} V_{e,i}^{\mu},$$

$$\mathcal{K} \ni \nabla_{\mu} \left( \zeta^{\mu} \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} - \sum_{i=1}^{2} f_{i} V_{e,i}^{\mu} \right),$$

$$(2.32)$$

where  $\tilde{P}^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu} - \frac{1}{\zeta^{\sigma}\zeta_{\sigma}}\zeta^{\mu}\zeta^{\nu}$ , and we have defined,

$$df_i = \frac{\alpha_{E,i}}{T}dT + T\alpha_{Q,i}d\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s.$$
 (2.33)

The actual computation is not neat and we have presented the details in appendix (B) for the aid of the readers interested in reproducing our results. Note that these constitutive relations are presented in terms of 'data' which are natural for this sector; readers can modify these to their favorite basis and get results which might look considerably messier. Moreover, these results are written in a particular 'hydrodynamic frame' chosen by aligning  $u^{\mu}$ , T,  $\mu$  along  $\beta^{\mu}$ ,  $\Lambda_{\beta}$ , which again can be modified according to reader's preference.

2. Let us now look at the parity-even solutions to eqn. (2.29) with  $\mathcal{N}^{\mu} = 0$ ,

$$0 = \frac{1}{2} \mathcal{T}^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} + \mathcal{J}^{\mu} \delta_{\mathcal{B}} A_{\mu} + \mathcal{K} \delta_{\mathcal{B}} \varphi + \mathcal{D}. \tag{2.34}$$

Every term in  $\mathcal{T}^{\mu\nu}$ ,  $\mathcal{J}^{\mu}$ ,  $\mathcal{K}$  must either cancel or contribute to  $\Delta$  which has to be a quadratic form. It follows that the terms in  $\mathcal{T}^{\mu\nu}$ ,  $\mathcal{J}^{\mu}$ ,  $\mathcal{K}$  must be proportional to  $\delta_{\mathcal{B}}g_{\mu\nu}$ ,  $\delta_{\mathcal{B}}A_{\mu}$ ,  $\delta_{\mathcal{B}}\varphi$ . Recall however that we have chosen to eliminate  $u^{\mu}\delta_{\mathcal{B}}g_{\mu\nu}$ ,  $u^{\mu}\delta_{\mathcal{B}}A_{\mu}$  using the equations of motion. For  $\Delta$  to be a quadratic form, it therefore implies that  $\mathcal{T}^{\mu\nu}$ ,  $\mathcal{J}^{\mu}$  cannot have a term like  $\#^{(\mu}u^{\nu)}$ ,  $\#u^{\mu}$  respectively for some vector  $\#^{\mu}$  and scalar #. With this input let us write down the most generic allowed form of the currents in terms of 20 new transport coefficients  $[\beta_{ij}]_{4\times4}$  (with  $\beta_{44}=\alpha/T$ ),  $[\kappa_{ij}]_{2\times2}$  and  $\eta$ ,

$$\mathcal{T}^{\mu\nu} \ni -T \left[ \left\{ \beta_{11} \tilde{P}^{\rho\sigma} + \beta_{12} \zeta^{\rho} \zeta^{\sigma} \right\} \tilde{P}^{\mu\nu} + \left\{ \beta_{21} \tilde{P}^{\mu\nu} + \beta_{22} \zeta^{\rho} \zeta^{\sigma} \right\} \zeta^{\mu} \zeta^{\nu} + 4\kappa_{11} \zeta^{(\mu} \tilde{P}^{\nu)(\rho} \zeta^{\sigma)} \right.$$

$$\left. + \eta \tilde{P}^{\mu\langle\rho} \tilde{P}^{\sigma\rangle\nu} \right] \frac{1}{2} \delta_{\mathcal{B}} g_{\rho\sigma} - T \left[ \beta_{13} \zeta^{\rho} \tilde{P}^{\mu\nu} + \beta_{23} \zeta^{\rho} \zeta^{\mu} \zeta^{\nu} + 2\kappa_{12} \zeta^{(\mu} \tilde{P}^{\nu)\rho} \right] \delta_{\mathcal{B}} A_{\rho}$$

$$- T \left[ \beta_{14} \tilde{P}^{\mu\nu} + \beta_{24} \zeta^{\mu} \zeta^{\nu} \right] \delta_{\mathcal{B}} \varphi,$$

$$= -\tilde{P}^{\mu\nu} \sum_{i=1}^{4} \beta_{1i} S_{i} - \zeta^{\mu} \zeta^{\nu} \sum_{i=1}^{4} \beta_{2i} S_{i} - 2\zeta^{(\mu} \sum_{i=1}^{2} \kappa_{1i} V_{i}^{\nu)} - \eta \sigma^{\mu\nu}. \tag{2.35}$$

$$\mathcal{J}^{\mu} \ni -T \left[ \left\{ \beta_{31} \tilde{P}^{\rho\sigma} + \beta_{32} \zeta^{\rho} \zeta^{\sigma} \right\} \zeta^{\mu} + 2\kappa_{21} \tilde{P}^{\mu(\rho} \zeta^{\sigma)} \right] \frac{1}{2} \delta_{\mathcal{B}} g_{\rho\sigma}$$

$$-T \left[ \beta_{33} \zeta^{\rho} \zeta^{\mu} + \kappa_{22} \tilde{P}^{\mu\rho} \right] \delta_{\mathcal{B}} A_{\rho} - T \left[ \beta_{34} \zeta^{\mu} \right] \delta_{\mathcal{B}} \varphi,$$

$$= -\zeta^{\mu} \sum_{i=1}^{4} \beta_{3i} S_{i} - \sum_{i=1}^{2} \kappa_{2i} V_{i}^{\mu}.$$

$$(2.36)$$

$$\mathcal{K} \ni -T \left[ \beta_{41} \tilde{P}^{\rho\sigma} + \beta_{42} \zeta^{\rho} \zeta^{\sigma} \right] \delta_{\mathcal{B}} g_{\rho\sigma} - T \left[ \beta_{43} \zeta^{\rho} \right] \delta_{\mathcal{B}} A_{\rho} = -\sum_{i=1}^{3} \beta_{4i} S_{i}. \tag{2.37}$$

Note that we did not include a term proportional to  $\delta_{\mathcal{B}}\varphi$  in  $\mathcal{K}$ , because such a term is already present in  $K = -\alpha\delta_{\mathcal{B}}\varphi + \nabla_{\mu}(R_s\xi^{\mu}) + \mathcal{K} + \mathcal{O}(\partial^2)$ . Defining  $\beta_{44} = \alpha/T$ , we can read out the parity-even quadratic form  $\Delta|_{even} = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{even}$ ,

$$T\Delta|_{even} = \sum_{i,j=1}^{4} S_i \beta_{ij} S_j + \sum_{i,j=1}^{2} V_i^{\mu} \kappa_{ij} V_{i,\mu} + \eta \sigma^{\mu\nu} \sigma_{\mu\nu},$$

$$= \sum_{i,j=1}^{4} S_i \beta_{(ij)} S_j + \sum_{i,j=1}^{2} V_i^{\mu} \kappa_{(ij)} V_{i,\mu} + \eta \sigma^{\mu\nu} \sigma_{\mu\nu}.$$
(2.38)

In the second step we have realized that only the symmetric parts of the matrices  $\beta_{ij}$  and  $\kappa_{ij}$  will survive in this expression, and will contribute towards dissipation. Thus only 14 out of 21 transport coefficients (including  $\alpha$ ) are dissipative; the remaining 7 are non-dissipative.

### 2.4.2 Parity Odd (4 Dimensions)

We can find the most general parity-odd solution of eqn. (2.29) in 3 steps: (1) first we consider a particular set of solutions which takes care of the anomaly  $N_H^{\perp}$  and proceed towards the non-anomalous constitutive relations, (2) then we write down the most general allowed parity-odd  $\mathcal{N}^{\mu}$  and find a set of constitutive relations pertaining to that, and (2) finally find the most general parity-odd constitutive relations with zero  $\mathcal{N}^{\mu}$ .

1. In 4 dimensions at the first order in the derivatives  $T_{\rm H}^{\mu\perp} = 0$  and  $J_{\rm H}^{\perp} = -\frac{3}{4}C^{(4)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ , which implies,

$$N_{\rm H}^{\perp} = -\frac{3}{4}\nu C^{(4)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{2.39}$$

A particular solution pertaining to eqn. (2.29) with this  $N_H^{\perp}$  is given as (see e.g. [23]),

$$\mathcal{T}^{\mu\nu} \ni 2\mu^{2}C^{(4)}u^{(\mu}\left(3B^{\nu)} + 2\mu\omega^{\nu}\right),$$

$$\mathcal{J}^{\mu} \ni \mu C^{(4)}\left(6B^{\mu} + 3\mu\omega^{\mu}\right),$$

$$\mathcal{K} \ni 0,$$

$$\mathcal{N}^{\mu} \ni \frac{\mu^{2}}{T}C^{(4)}\left(3B^{\mu} + \mu\omega^{\mu}\right).$$
(2.40)

Here we have defined the magnetic field and fluid vorticity as,

$$B^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_{\nu} F_{\rho\sigma}, \qquad \omega^{\mu} = \epsilon^{\mu\nu\rho\sigma} u_{\nu} \partial_{\rho} u_{\sigma}. \tag{2.41}$$

2. One can check that the most general form of  $\mathcal{N}^{\mu}$  (whose divergence only contains product of derivatives and has at least one  $\delta_{\mathcal{B}}$  per term) can be written as,

$$\mathcal{N}^{\mu} = g_1 \left( \beta^{\mu} \tilde{S}_{e,1} + \tilde{V}_3^{\mu} \right) + g_2 \left( \beta^{\mu} \tilde{S}_{e,2} + \tilde{V}_2^{\mu} \right) + C_1 T^2 \omega^{\mu}, \tag{2.42}$$

where g's are functions of T,  $\nu$ ,  $\hat{\mu}_s$ , and  $C_1$  is a constant. From here we can directly read out the corresponding constitutive relations,

$$\mathcal{T}^{\mu\nu} \ni u^{\mu}u^{\nu} \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} + \left( \zeta^{\mu} \zeta^{\nu} - 2(u^{\rho} \xi_{\rho}) u^{(\mu} \zeta^{\nu}) \right) \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} - \zeta^{\mu} \zeta^{\nu} \sum_{i=1}^{2} \frac{1}{2\hat{\mu}_{s}} g_{i} \tilde{S}_{e,i} \\
- 2u^{(\mu} \sum_{i=1}^{2} g_{i} \tilde{V}_{e,2+i}^{\nu)} - u^{(\mu} \left( 2P_{\alpha}^{\nu)} - u^{\nu} u_{\alpha} \right) \epsilon^{\alpha\rho\sigma\tau} \nabla_{\sigma} \left( Tg_{1}u_{\tau}\zeta_{\rho} \right) + 4C_{1}T^{3}\omega^{(\mu}u^{\nu)} \\
\mathcal{J}^{\mu} \ni u^{\mu} \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{e,i} - \zeta^{\mu} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} + \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{\mu} + \epsilon^{\mu\nu\rho\sigma} \nabla_{\nu} \left( Tg_{2}\zeta_{\rho}u_{\sigma} \right), \\
\mathcal{K} \ni \nabla_{\mu} \left( \zeta^{\mu} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{\mu} \right), \tag{2.43}$$

where we have defined,

$$dg_i = \frac{\tilde{\alpha}_{E,i}}{T}dT + T\tilde{\alpha}_{Q,i}d\nu + \left(\tilde{\alpha}_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s.$$
 (2.44)

The actual computation is not neat and we have presented the details in appendix (B) for interested readers.

3. We should finally consider the parity-odd constitutive relations that satisfy eqn. (2.29) with zero LHS. Following our discussion in the parity-even sector, the allowed form of the constitutive relations can be written down in terms of 5 coefficients  $[\tilde{\kappa}_{ij}]_{2\times 2}$  and  $\tilde{\eta}$ ,

$$\mathcal{T}^{\mu\nu} \ni -Tu_{\tau}\zeta_{\kappa} \left[ 4\tilde{\kappa}_{11}\zeta^{(\mu}\epsilon^{\nu)\tau\kappa(\rho}\zeta^{\sigma)} + \tilde{\eta}\tilde{P}^{\lambda(\mu}\epsilon^{\nu)\tau\kappa(\rho}\tilde{P}^{\sigma)}_{\lambda} \right] \frac{1}{2}\delta_{\mathbb{B}}g_{\rho\sigma} - Tu_{\tau}\zeta_{\kappa} \left[ 2\tilde{\kappa}_{12}\zeta^{(\mu}\epsilon^{\nu)\tau\kappa\rho} \right] \delta_{\mathbb{B}}A_{\rho},$$

$$= -2\zeta^{(\mu}\sum_{i=1}^{2} \tilde{\kappa}_{1i}\tilde{V}_{i}^{\nu)} - \tilde{\eta}\tilde{\sigma}^{\mu\nu},$$

$$\mathcal{J}^{\mu} \ni -Tu_{\tau}\zeta_{\kappa} \left[ 2\tilde{\kappa}_{21}\epsilon^{\mu\tau\kappa(\rho}\zeta^{\sigma)} \right] \frac{1}{2}\delta_{\mathbb{B}}g_{\rho\sigma} - Tu_{\tau}\zeta_{\kappa} \left[ \tilde{\kappa}_{22}\epsilon^{\mu\tau\kappa\rho} \right] \delta_{\mathbb{B}}A_{\rho},$$

$$\ni -\sum_{i=1}^{2} \tilde{\kappa}_{2i}\tilde{V}_{i}^{\mu},$$

$$\mathcal{K} \ni 0. \tag{2.45}$$

One can check that these constitutive relations trivially satisfy eqn. (2.29) with zero LHS and the quadratic form  $\Delta|_{odd} = \mathcal{D}|_{odd}$  is given as,

$$T\Delta|_{odd} = \epsilon^{\mu\nu\tau\kappa} u_{\tau} \zeta_{\kappa} \left[ \sum_{i=1}^{2} V_{i,\mu} \tilde{\kappa}_{ij} V_{j,\nu} + \tilde{\eta} \sigma_{\mu\rho} \sigma_{\nu}^{\rho} \right]$$
$$= \epsilon^{\mu\nu\tau\kappa} u_{\tau} \zeta_{\kappa} \sum_{i=1}^{2} V_{i,\mu} \tilde{\kappa}_{[ij]} V_{j,\nu} = 2 \epsilon^{\mu\nu\tau\kappa} u_{\tau} \zeta_{\kappa} V_{1,\mu} \tilde{\kappa}_{[12]} V_{2,\nu}. \tag{2.46}$$

It follows that out of the 5 transport coefficients, only 1 contribute to dissipation and the other 4 are non-dissipative.

### 2.4.3 Positivity Constraints

The dissipative transport coefficients are required to satisfy a set of inequalities to satisfy  $\Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{even} + \mathcal{D}|_{odd} \geq 0$ ,

$$T\Delta = \sum_{i,j=1}^{4} S_i \beta_{(ij)} S_j + \left( \sum_{i,j=1}^{2} V_i^{\mu} \kappa_{(ij)} V_{i,\mu} + \sum_{i=1}^{2} V_i^{\mu} \tilde{\kappa}_{[ij]} \tilde{V}_{j,\mu} \right) + \eta \sigma^{\mu\nu} \sigma_{\mu\nu}. \tag{2.47}$$

We want this expression to be a quadratic form, which it nearly is except the parity-odd term in the brackets. However this term can be made into a quadratic form by noticing that the square of a parity odd term is parity-even, due to the identity,

$$(\epsilon^{\mu\nu\rho\sigma}u_{\rho}\zeta_{\sigma})(\epsilon_{\tau\nu\alpha\beta}u^{\alpha}\zeta^{\beta}) = \tilde{P}^{\mu}_{\tau}\zeta^{\nu}\zeta_{\nu} = -2\hat{\mu}_{s}\tilde{P}^{\mu}_{\tau}. \tag{2.48}$$

We define,

$$\begin{pmatrix} V_1'^{\mu} \\ V_2'^{\mu} \end{pmatrix} = \begin{pmatrix} V_1^{\mu} \\ V_2^{\mu} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^{\mu} \\ \tilde{V}_2^{\mu} \end{pmatrix}, \qquad \kappa'_{ij} = \kappa_{ij} + k_{ij}, \qquad k_{[ij]} = 0, \tag{2.49}$$

such that,

$$\sum_{i,j=1}^{2} V_{i}^{\prime \mu} \kappa_{(ij)}^{\prime} V_{i,\mu}^{\prime} = \sum_{i,j=1}^{2} V_{i}^{\mu} \kappa_{(ij)} V_{i,\mu} + \sum_{i=1}^{2} V_{i}^{\mu} \tilde{\kappa}_{[ij]} \tilde{V}_{j,\mu}.$$
 (2.50)

Using the identity eqn. (2.48), the above equation can be easily solved to give,

$$a_{12} = \frac{\tilde{\kappa}_{12}^{(a)}}{\kappa_{11}}, \qquad k_{1i} = k_{i1} = 0, \qquad k_{22} = 2\hat{\mu}_s \frac{\tilde{\kappa}_{[12]}}{\kappa_{11}}.$$
 (2.51)

Consequently  $\Delta$  will take the form,

$$T\Delta = \sum_{i,j=1}^{4} S_i \beta_{(ij)} S_j + \sum_{i,j=1}^{2} V_i^{\prime \mu} \kappa_{(ij)}^{\prime} V_{i,\mu}^{\prime} + \eta \sigma^{\mu\nu} \sigma_{\mu\nu}.$$
 (2.52)

Given  $T \geq 0$ , the condition  $\Delta \geq 0$  implies that  $\eta \geq 0$  and the matrices  $[\beta_{(ij)}]_{4\times 4}$ ,  $[\kappa'_{(ij)}]_{2\times 2}$  have all non-negative eigenvalues. This gives 7 inequalities among 15 dissipative transport coefficients, and 8 are completely arbitrary.

### 2.5 Summary

We have completed the analysis of a superfluid up to the first order in derivatives. Here we summarize the results. We found that the entire superfluid transport up to the first order in derivatives is characterized by an ideal order pressure P, 30 first order transport coefficients which are functions of T,  $\mu/T$ ,  $\hat{\mu}_s$ , and two constants  $C_1$ ,  $C^{(4)}$ . P,  $C_1$  and  $C^{(4)}$  along with 4 transport coefficients,

Parity Even (2): 
$$f_1$$
,  $f_2$ , Parity Odd (2):  $g_1$ ,  $g_2$ , (2.53)

totally determine the *hydrostatic* transport (part of the constitutive relations that survive at equilibrium). *Non-hydrostatic non-dissipative* transport (part that does not survive at equilibrium but doesn't contribute to  $\Delta \geq 0$  either) is given by 11 transport coefficients,

Parity Even (7): 
$$[\beta_{[ij]}]_{4\times4}$$
 (antisymmetric),  $[\kappa_{[ij]}]_{2\times2}$  (antisymmetric),

Parity Odd (4): 
$$[\tilde{\kappa}_{(ij)}]_{2\times 2}$$
 (symmetric),  $\tilde{\eta}$ . (2.54)

Finally the entire dissipative transport is given by 15 transport coefficients,

Parity Even (14): 
$$[\beta_{(ij)}]_{4\times4}$$
 (symmetric),  $[\kappa_{(ij)}]_{2\times2}$  (symmetric),  $\eta$ ,

Parity Odd (1): 
$$\left[\tilde{\kappa}_{[ij]}\right]_{2\times2}$$
 (antisymmetric). (2.55)

These dissipative transport coefficients follow a set of inequalities ( $\kappa'$  is defined in eqn. (2.49)),

$$[\beta_{(ij)}]_{4\times 4}, \quad [\kappa'_{(ij)}]_{2\times 2}, \quad \eta \geq 0,$$
 (2.56)

where a 'non-negative matrix' implies all its eigenvalues are non-negative. Using P,  $f_i$ ,  $g_i$  we define some new functions,

$$dP = SdT + Qd\mu + R_s d\mu_s, \qquad E + P = ST + Q\mu,$$

$$df_i = \frac{\alpha_{E,i}}{T}dT + T\alpha_{Q,i}d\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s, \qquad \alpha_{E,i} + f_i = \alpha_{S,i}T + \alpha_{Q,i}\mu,$$

$$dg_i = \frac{\tilde{\alpha}_{E,i}}{T}dT + T\tilde{\alpha}_{Q,i}d\nu + \left(\tilde{\alpha}_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s, \qquad \tilde{\alpha}_{E,i} + g_i = \tilde{\alpha}_{S,i}T + \tilde{\alpha}_{Q,i}\mu. \tag{2.57}$$

In terms of these transport coefficients, corrections to the Josephson equation (K = 0) coming from the first order superfluid transport are given as (here  $\beta_{44} = \alpha/T$ ),

$$u^{\mu}\xi_{\mu} - \mu = \frac{1}{\beta_{44}} \nabla_{\mu} (R_{s}\xi^{\mu}) - \sum_{i=1}^{3} \frac{\beta_{4i}}{\beta_{44}} S_{i} + \frac{1}{\beta_{44}} \nabla_{\mu} \left( \zeta^{\mu} \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} + \zeta^{\mu} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \sum_{i=1}^{2} f_{i} V_{e,i}^{\mu} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{\mu} \right) + \mathcal{O}(\partial^{2}), \quad (2.58)$$

which can be seen as determining  $u^{\mu}\xi_{\mu}$  in terms of the other superfluid variables. Note that though this equation contains second order terms, it is only correct up to the first order in derivatives, and will admit further corrections coming from higher order superfluid transport. The energy-momentum tensor, charge current and entropy current up to first order in derivatives are however given as,

$$T^{\mu\nu} = (E+P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + R_{s}\xi^{\mu}\xi^{\nu} + \mathcal{T}^{\mu\nu} + \mathcal{O}(\partial^{2}),$$

$$J^{\mu} = Qu^{\mu} - R_{s}\xi^{\mu} + \mathcal{J}^{\mu} + \mathcal{O}(\partial^{2}),$$

$$J^{\mu}_{S} = Su^{\mu} + \mathcal{S}^{\mu} + \mathcal{O}(\partial^{2}),$$
(2.59)

where the higher derivative corrections are,

$$\mathcal{T}^{\mu\nu} = u^{\mu} u^{\nu} \left[ \sum_{i=1}^{2} \alpha_{E,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{\sigma} (T f_{1} \zeta^{\sigma}) + \epsilon^{\alpha \rho \sigma \tau} u_{\alpha} \nabla_{\rho} (T g_{1} u_{\sigma} \zeta_{\tau}) \right] 
+ 2u^{(\mu} \zeta^{\nu)} \left[ \sum_{i=1}^{2} f_{i} S_{4+i} - (u^{\rho} \xi_{\rho}) \left( \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} \right) + \frac{1}{2\hat{\mu}_{s}} \epsilon^{\alpha \rho \sigma \tau} \zeta_{\alpha} \nabla_{\rho} (T g_{1} u_{\sigma} \zeta_{\tau}) \right] 
+ \zeta^{\mu} \zeta^{\nu} \left[ \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \left( \tilde{\alpha}_{R_{s},i} - \frac{g_{i}}{2\hat{\mu}_{s}} \right) \tilde{S}_{e,i} - \sum_{i=1}^{4} \beta_{2i} S_{i} \right] 
+ 2u^{(\mu} \left[ (\xi^{\sigma} u_{\sigma}) \sum_{i=1}^{2} f_{i} V_{e,i}^{\nu} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,2+i}^{\nu} - \tilde{P}_{\alpha}^{\nu} \epsilon^{\alpha \rho \sigma \tau} \nabla_{\rho} (T g_{1} u_{\sigma} \zeta_{\tau}) + 2C_{1} T^{3} \omega^{\nu} \right) 
+ C^{(4)} \mu^{2} \left( 3B^{\nu} + 2\mu \omega^{\nu} \right) \right] - 2\zeta^{(\mu} \left[ \sum_{i=1}^{2} f_{i} V_{e,i}^{\nu} + \sum_{i=1}^{2} \kappa_{1i} V_{i}^{\nu} + \sum_{i=1}^{2} \tilde{\kappa}_{1i} \tilde{V}_{i}^{\nu} \right] 
+ \tilde{P}^{\mu\nu} \left[ \sum_{i=1}^{2} f_{i} S_{e,i} - \sum_{i=1}^{4} \beta_{1i} S_{i} \right] - \eta \sigma^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu}, \tag{2.60}$$

$$\mathcal{J}^{\mu} = u^{\mu} \left[ \sum_{i=1}^{2} \alpha_{Q,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{\nu} (T f_{2} \zeta^{\nu}) + \epsilon^{\alpha \nu \rho \sigma} u_{\alpha} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma}) \right] 
- \zeta^{\mu} \left[ \sum_{i=1}^{2} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} + \sum_{i=1}^{4} \beta_{3i} S_{i} - \frac{1}{2\hat{\mu}_{s}} \epsilon^{\alpha \nu \rho \sigma} \zeta_{\alpha} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma}) \right] 
+ \sum_{i=1}^{2} f_{i} V_{e,i}^{\mu} + \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{\mu} - \sum_{i=1}^{2} \kappa_{2i} V_{i}^{\mu} - \sum_{i=1}^{2} \tilde{\kappa}_{2i} \tilde{V}_{i}^{\mu} 
- \tilde{P}_{\alpha}^{\mu} \epsilon^{\alpha \nu \rho \sigma} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma}) + 3\mu C^{(4)} (2B^{\mu} + \mu \omega^{\mu}), \qquad (2.61)$$

$$S^{\mu} = g_{1} \frac{1}{T} \epsilon^{\mu\nu\rho\sigma} u_{\nu} \zeta_{\rho} \partial_{\sigma} T + g_{2} T \epsilon^{\mu\nu\rho\sigma} u_{\nu} \xi_{\rho} \partial_{\sigma} \nu + 3C_{1} T^{2} \omega^{\mu}$$

$$+ u^{\mu} \left[ \sum_{i=1}^{2} \alpha_{S,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^{2}} \nabla_{\sigma} (T f_{1} \zeta^{\sigma}) + \frac{\mu}{T^{2}} \nabla_{\nu} (T f_{2} \zeta^{\nu}) \right]$$

$$+ \frac{1}{T} \epsilon^{\alpha\nu\rho\sigma} u_{\alpha} \nabla_{\nu} (T g_{1} u_{\rho} \zeta_{\sigma}) - \frac{\mu}{T} \epsilon^{\alpha\rho\sigma\tau} u_{\alpha} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma})$$

$$+ \frac{1}{T} \zeta^{\mu} \left[ \sum_{i=1}^{4} \mu \beta_{3i} S_{i} + \frac{1}{2\hat{\mu}_{s}} \epsilon^{\alpha\rho\sigma\tau} \zeta_{\alpha} \nabla_{\rho} (T g_{1} u_{\sigma} \zeta_{\tau}) - \frac{\mu}{2\hat{\mu}_{s}} \zeta_{\alpha} \epsilon^{\alpha\nu\rho\sigma} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma}) \right]$$

$$+ \frac{\mu}{T} \sum_{i=1}^{2} \kappa_{2i} V_{i}^{\mu} + \frac{\mu}{T} \sum_{i=1}^{2} \tilde{\kappa}_{2i} \tilde{V}_{i}^{\mu} - \frac{1}{T} \tilde{P}^{\mu}_{\alpha} \epsilon^{\alpha\nu\rho\sigma} \nabla_{\nu} (T g_{1} u_{\rho} \zeta_{\sigma}) + \frac{\mu}{T} \tilde{P}^{\mu}_{\alpha} \epsilon^{\alpha\nu\rho\sigma} \nabla_{\nu} (T g_{2} u_{\rho} \zeta_{\sigma}).$$

$$(2.62)$$

The scalar  $S_4 = T\delta_{\mathcal{B}}\varphi = u^{\mu}\xi_{\mu} - \mu$  appearing here can be eliminated in favor of  $\nabla_{\mu}(R_s\xi^{\mu})$  using the Josephson equation. We will like to reiterate that these results are presented in a particular hydrodynamic frame (gained by aligning  $u^{\mu}$ , T,  $\mu$  along  $\beta^{\mu}$ ,  $\Lambda_{\beta}$ ) and in a "natural" choice of basis for the independent data. They can be transformed to any other preferred hydrodynamic frame or basis by a straight forward substitution.

In deriving these constitutive relations, we have only used the second law of thermodynamics. To compare these results with the existing literature [11–13], one might need to further filter these results with requirements like microscopic reversibility (Onsager relations), time reversal invariance and CPT invariance. For example, Onsager relations are known to turn off 7 parity-even non-dissipative coefficients  $[\beta_{[ij]}]_{4\times4}$ ,  $[\kappa_{[ij]}]_{2\times2}$  and the only parity-odd dissipative coefficient  $[\tilde{\kappa}_{[ij]}]_{2\times2}$  [11]. To avoid confusion, also note that there is a coefficient  $f_3$  appearing in eqn. (2.30) which we removed by using the  $\varphi$  equation of motion (or equivalently, by redefining  $\varphi$ ). This coefficient has been included in the counting of independent transport coefficients in [12].

# 3 | Null Superfluids

In [16] we proposed "null fluids" as a new viewpoint of Galilean fluids. In this section, we will further extend this formalism to include Galilean superfluids. The main benefit of working with "null (super)fluids" is that it is a "relativistic embedding" of Galilean (super)fluids into one higher dimension and enables us to directly use the existing relativistic machinery to read

out the respective Galilean results. In this sense, our in-depth review of relativistic superfluids in the previous section will be vital for our discussion of null/Galilean superfluids. To make the transition from relativistic  $\rightarrow$  null  $\rightarrow$  Galilean superfluids manifest, we will step by step imitate our relativistic discussion of the previous section with appropriate accommodations for null superfluids. Later in §4, we will translate our null superfluid results to the better known Newton-Cartan and conventional non-covariant notations.

### 3.1 Null Backgrounds and Null Superfluids

Let us quickly recap null backgrounds [15, 16], which are a natural 'embedding' of Galilean (Newton-Cartan) backgrounds into a relativistic spacetime of one higher dimension. Consider a (d+1)-dimensional manifold  $\mathcal{M}_{(d+1)}$  equipped with a metric  $g_{MN}$  and a U(1) gauge field  $A_M$ . Infinitesimal diffeomorphisms and gauge transformation with parameters  $\mathcal{X} = \{\chi^M, \Lambda_\chi\}$  respectively, act on these background fields as,

$$\delta \chi g_{MN} = \nabla_M \chi_N + \nabla_N \chi_M, \qquad \delta \chi A_M = \partial_M (\Lambda_\chi + \chi^N A_N) + \chi^N F_{NM}. \tag{3.1}$$

The characteristic feature of a null background is the existence of a compatible null isometry  $\mathcal{V} = \{V^M, \Lambda_V\}$  which satisfies:  $V^M V_M = 0$ ,  $\nabla_M V^N = 0$ ,  $V^M A_M + \Lambda_V = -1$  and,

$$\delta \gamma g_{MN} = \nabla_M V_N + \nabla_N V_M = 0, \qquad \delta \gamma A_M = \partial_M (\Lambda_V + V^N A_N) + V^N F_{NM} = V^N F_{NM} = 0. \quad (3.2)$$

Since we will be interested in studying superfluids on this background, we introduce a preferred U(1) phase  $\varphi$  which transforms under diffeomorphisms and infinitesimal gauge transformations as  $\delta_{\chi}\varphi = \chi^{M}\partial_{M}\varphi - \Lambda_{\chi}$ . The covariant derivative of  $\varphi$  is known as the *superfluid velocity*,

$$\xi_M = \partial_M \varphi + A_M. \tag{3.3}$$

We require  $\varphi$  to respect the null isometry  $\mathcal{V}$ , i.e.  $\delta_{\mathcal{V}}\varphi = V^M\partial_M\varphi - \Lambda_V = V^M\xi_M - 1 = 0$ , which implies  $V^M\xi_M = -1$ . The remainder of the story is exactly same as the relativistic case: any theory coupled to a null background has an energy-momentum tensor  $T^{MN}$  and a charge current  $J^M$  in its spectrum. The respective conservation laws are given as,

$$\nabla_{M} T^{MN} = F^{NR} J_{R} + \Gamma_{H}^{N\perp} + \xi^{M} K, \qquad \nabla_{M} J^{M} = J_{H}^{\perp} - K, \tag{3.4}$$

where,

$$K = 0, (3.5)$$

is the  $\varphi$  equation of motion. Since eqns. (3.4) and (3.5) are (d+3) equations in (d+1) dimensions, they can provide dynamics for a superfluid described by an arbitrary set of (d+2) variables in addition to the phase  $\varphi$ . We choose these to be a normalized null fluid velocity  $u^M$  (with  $u^M V_M = -1$ ,  $u^M u_M = 0$ ), a temperature T, a mass chemical potential  $\mu_n$ , and a chemical potential  $\mu_n$ , known as the hydrodynamic fields. A null superfluid hence is completely

<sup>&</sup>lt;sup>4</sup>This condition can be thought of as fixing a component of the (d+1)-dimensional gauge field  $A_M$ , leaving it with only d independent components mapping bijectively to the d-dimensional Galilean gauge field. As opposed to the null backgrounds defined in [16] where we set  $V^M A_M + \Lambda_V = 0$ , for superfluids we realize that it is more suitable to fix  $V^M A_M + \Lambda_V = -1$  instead.

characterized by gauge-invariant expressions of  $T^{MN}$ ,  $J^{M}$ , K in terms of  $g_{MN}$ ,  $A_{M}$ ,  $u^{M}$ , T,  $\mu_{n}$ ,  $\mu$  and  $\xi_{M}$ , known as the *null superfluid constitutive relations*. The near equilibrium assumption allows us to arrange these constitutive relations as a perturbative expansion in derivatives (known as the *derivative* or *gradiant expansion*).

Same as the relativistic case, null superfluid is also required to satisfy a version of the second law of thermodynamics. It states that there must exist an entropy current  $J_S^M$  whose divergence is positive semi-definite everywhere, i.e.,

$$\nabla_M J_S^M \ge 0, \tag{3.6}$$

as long as the superfluid is thermodynamically isolated (i.e. conservation laws eqn. (3.4) are satisfied), irrespective of K being zero. The job of null superfluid dynamics now is to find the most general constitutive relations  $T^{MN}$ ,  $J^M$ , K and an associated  $J_S^M$ ,  $\Delta$  order by order in derivative expansion, such that eqn. (3.6) is satisfied for thermodynamically isolated fluids. Owing to our previous experiences with the second law however, we switch to the offshell formalism in the next subsection for simplicity.

### 3.2 Offshell Formalism for Null (Super)fluids

We couple the fluid to an external momentum  $P_{ext}^{M}$  and charge  $Q_{ext}$  source, so that the conservation laws are no longer satisfied. Having done that, the second law eqn. (3.6) will be modified with an arbitrary combination of the conservation laws to get,

$$\nabla_{M} J_{S}^{M} + \beta_{N} \left( \nabla_{M} T^{MN} - F^{NR} J_{R} - \mathbf{T}_{H}^{N\perp} - \xi^{M} K \right)$$

$$+ \left( \Lambda_{\beta} + A_{M} \beta^{M} \right) \left( \nabla_{M} J^{M} - \mathbf{J}_{H}^{\perp} + K \right) = \Delta \geq 0, \quad (3.7)$$

where  $\mathcal{B} = \{\beta^M, \Lambda_\beta\}$  are some arbitrary fields. Recall that the hydrodynamic fields  $u^M$ , T,  $\mu_n$ ,  $\mu$  were some arbitrary (d+2) fields chosen to describe the fluid. Like in any field theory, they are permitted to admit an arbitrary redefinition among themselves without changing the physics. This huge amount of freedom can be fixed by explicitly choosing,

$$u^{M} = -\frac{\beta^{M}}{V_{M}\beta^{M}} + \frac{\beta^{R}\beta_{R}V^{M}}{2(V_{N}\beta^{N})^{2}}, \quad T = -\frac{1}{V_{M}\beta^{M}}, \quad \mu_{n} = \frac{\beta^{M}U_{M}}{2(V_{N}\beta^{N})^{2}}, \quad \mu = -\frac{\Lambda_{\beta} + A_{M}\beta^{M}}{V_{N}\beta^{N}}. \quad (3.8)$$

or conversely,

$$\beta^{M} = \frac{1}{T} (u^{M} - \mu_{n} V^{M}), \qquad \Lambda_{\beta} = \frac{\mu}{T} - A_{M} u^{M}.$$
 (3.9)

We define a free energy current,

$$-\frac{G^{M}}{T} = N^{M} = S^{M} + T^{MN}\beta_{N} + (\Lambda_{\beta} + \beta^{N}A_{N})J^{M}, \quad -\frac{G_{H}^{M}}{T} = N_{H}^{\perp} = \beta_{M}T_{H}^{M\perp} + (\Lambda_{\beta} + \beta^{M}A_{M})J_{H}^{\perp},$$
(3.10)

which turns the offshell second law in eqn. (3.7) to a free energy conservation equation,

$$\nabla_{M} N^{M} - N_{H}^{\perp} = \frac{1}{2} T^{MN} \delta_{\mathcal{B}} g_{MN} + J^{M} \delta_{\mathcal{B}} A_{M} + K \delta_{\mathcal{B}} \varphi + \Delta, \qquad \Delta \ge 0.$$
 (3.11)

Now similar to our analysis of relativistic superfluids, we will try to find the most generic  $T^{MN}$ ,  $J^{M}$ , K in terms of  $g_{MN}$ ,  $A_{M}$ ,  $\beta^{M}$ ,  $\Lambda_{\beta}$ ,  $\varphi$  which solves this equation for some  $N^{M}$ ,  $\Delta$ . Again

however, these expressions will be shy of being the null superfluid constitutive relations because of their dependence on the external sources  $P_{ext}^M$ ,  $Q_{ext}$ . To fix this, we will only consider the expressions for  $T^{MN}$ ,  $J^M$ , K which are independent of certain data that can be eliminated using the conservation laws. For the most part, the following analysis along with the wordings would be exactly same as has been used in the previous section for relativistic superfluids, except for certain modifications to accommodate the compatible null isometry.

### 3.2.1 Josephson Equation for Null (Galilean) Superfluids

In the study of superfluids, the U(1) phase  $\varphi$  is generally taken to be order -1 in the derivative expansion, while its covariant derivative  $\xi_M$  is taken to be order 0. The reason being that the true dynamical degrees of freedom are encoded in the fluctuations of  $\varphi$  along the U(1) circle, and not in  $\varphi$  itself. It implies that the  $K\delta_{\mathcal{B}}\varphi$  term in the free energy conservation eqn. (3.11) is allowed to be order zero, if K has an order 0 term. This gives us the unique solution to eqn. (3.11) at zero derivative order,

$$N^{M}, T^{MN}, J^{M} = \mathcal{O}(\partial^{0}), \quad K = -\alpha \delta_{\mathcal{B}} \varphi + \mathcal{O}(\partial), \quad \Delta = \alpha (\delta_{\mathcal{B}} \varphi)^{2} + \mathcal{O}(\partial),$$
 (3.12)

for some "transport coefficient"  $\alpha \geq 0$ . Note that the  $\varphi$  equation of motion at this order will read  $K = -\alpha \delta_{\mathbb{B}} \varphi + \mathcal{O}(\partial) = 0$ , implying,

$$\delta_{\mathcal{B}}\varphi = \frac{1}{T} \left( u^{M} \xi_{M} + \mu_{n} - \mu \right) = \mathcal{O}(\partial) \quad \Longrightarrow \quad u^{M} \xi_{M} = \mu - \mu_{n} + \mathcal{O}(\partial). \tag{3.13}$$

This is the Josephson equation for null superfluids. This condition also ensures that  $\Delta$  is at least  $\mathcal{O}(\partial)$ , avoiding "ideal superfluid dissipation". From this point onward, it would be beneficial to think of  $\delta_{\mathcal{B}}\varphi$  as an order 1 data in derivative expansion rather than 0.

#### 3.3 Ideal Null Superfluids

Let us now move on to the ideal null superfluids, i.e. null superfluid constitutive relations that satisfy the free energy conservation eqn. (3.11) at first derivative order. At ideal order, the most generic tensorial form of various quantities appearing in eqn. (3.11) can be written as,

$$T^{MN} = R_n u^M u^N + 2E u^{(M} V^{N)} + P P^{MN} + R_s \xi^M \xi^N + 2\lambda_1 \xi^{(M} V^{N)} + 2\lambda_2 \xi^{(M} u^N) + R_v V^M V^M + \mathcal{O}(\partial),$$

$$J^M = Q u^M + Q_s \xi^M + Q_v V^M + \mathcal{O}(\partial),$$

$$K = -\alpha \delta_B \varphi + K_{ideal} + \mathcal{O}(\partial),$$

$$N^M = N u^M + N_s \xi^M + N_v V^M + \mathcal{O}(\partial),$$

$$\Delta = (\alpha \delta_B \varphi)^2 + \Delta_{ideal} + \mathcal{O}(\partial^2),$$
(3.14)

where  $R_n$ , E, P,  $R_s$ ,  $\lambda_1$ ,  $\lambda_2$ , Q,  $Q_s$ ,  $K_{ideal}$ , N,  $N_s$  are functions of T,  $\mu$ ,  $\mu_n$  and  $\mu_s \equiv -\frac{1}{2} \xi^M \xi_M$ . We have omitted the only other possible scalar  $\delta_{\mathcal{B}} \varphi$  in the functional dependence, because using the  $\varphi$  equation of motion we know that it is no longer an independent quantity. The coefficients  $R_v$ ,  $Q_v$ ,  $N_v$  do not contain any physical information, because their contribution to

the conservation laws trivially vanish owing to V being an isometry. Plugging eqn. (3.14) in eqn. (3.11) we can find,

$$(Q_{s} + R_{s})\xi^{M} \left(\nabla_{M}\nu + \frac{1}{T}u^{N}F_{NM}\right) + \frac{\lambda_{1}}{T^{2}}\xi^{M}\nabla_{M}T + \lambda_{2}\xi^{N} \left(\nabla_{N}\nu_{n} + u^{M}\nabla_{M}U_{N}\right)$$

$$\nabla_{M} \left(\left(\frac{P}{T} - N\right)u^{N}\right) + \frac{1}{T}u^{\mu} \left(\nabla_{\mu}E - T\nabla_{M}S - \mu_{n}\nabla_{M}R_{n} - \mu\nabla_{M}Q + R_{s}\nabla_{N}\mu_{s}\right)$$

$$+ \nabla_{M} \left(\left(\delta_{\mathcal{B}}\varphi R_{s} - N_{s}\right)\xi^{M}\right) + \left(K_{ideal} - \nabla_{M}(R_{s}\xi^{M})\right)\delta_{\mathcal{B}}\varphi + \Delta_{ideal} = 0, \quad (3.15)$$

where we have defined S through the "Euler equation",

$$E + P = ST + Q\mu + R_n\mu_n. \tag{3.16}$$

Eqn. (3.15) will imply a set of relations among various coefficients,

$$Q_s = -R_s, \quad \lambda_1 = \lambda_2 = 0, \quad N = \frac{P}{T}, \quad N_s = \delta_{\mathcal{B}} \varphi R_s, \quad K_{ideal} = \nabla_M (R_s \xi^M), \quad \Delta_{ideal} = 0,$$

$$(3.17)$$

and the "first law of thermodynamics",

$$dE = TdS + \mu dQ + \mu_n dR_n - R_s d\mu_s, \tag{3.18}$$

giving physical meaning to the quantities we have introduced in eqn. (3.14). Finally, we have the full set of null superfluid constitutive relations up to ideal order satisfying the second law,

$$T^{MN} = R_n u^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + R_s \xi^M \xi^N + R_v V^M V^N + \mathcal{O}(\partial),$$

$$J^M = Qu^M - R_s \xi^M + Q_v V^M + \mathcal{O}(\partial),$$

$$K = -\alpha \delta_{\mathcal{B}} \varphi + \nabla_M (R_s \xi^M) + \mathcal{O}(\partial),$$

$$N^M = \frac{P}{T} u^M + \delta_{\mathcal{B}} \varphi R_s \xi^M + N_v V^M + \mathcal{O}(\partial),$$

$$J_S^M = N^M - \frac{1}{T} (T^{MN} u_N - \mu_n T^{MN} V_N + \mu J^M) = Su^M + S_v V^M + \mathcal{O}(\partial). \tag{3.19}$$

Here  $S_v = N_v + \frac{1}{T} (R_v - \mu_n E - \mu Q_v)$ , which again doesn't contain any physical information. These are the ideal null superfluid constitutive relations. Note that we have included first order terms in K,  $N^M$  which can be ignored when talking about the ideal order, but are required for internal consistency with eqn. (3.11). The  $\varphi$  equation of motion K = 0 will imply,

$$\alpha \delta_{\mathcal{B}} \varphi = \nabla_{M}(R_{s} \xi^{M}) + \mathcal{O}(\partial) \quad \Longrightarrow \quad u^{M} \xi_{M} = \mu - \mu_{n} + \frac{T}{\alpha} \nabla_{M}(R_{s} \xi^{M}) + \mathcal{O}(\partial), \tag{3.20}$$

which is a first order correction to the Josephson equation. Note however that this equation can admit further one derivative corrections due to the first order constitutive relations discussed in the next subsection; the correction mentioned here is only how the ideal null superfluid transport affects the Josephson equation. The conservation laws on the other hand are complete up to the first order in derivatives,

$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}\left(T(E+P)V_{M}+RTu_{M}\right)\right)+QT\delta_{\mathcal{B}}A_{M}=-\xi_{M}\alpha\delta_{\mathcal{B}}\varphi+\mathcal{O}(\partial^{2}),$$

$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}QT\right)=\alpha\delta_{\mathcal{B}}\varphi+\mathcal{O}(\partial^{2}).$$
(3.21)

These equations provide a set of relations between  $\delta_{\mathcal{B}}\varphi$ ,  $\delta_{\mathcal{B}}g_{MN}$  and  $\delta_{\mathcal{B}}A_{M}$ , which can be used to eliminate a vector  $u^{M}\delta_{\mathcal{B}}g_{MN}$  and a scalar  $u^{M}\delta_{\mathcal{B}}A_{M}$  (see table (3)) from the first order null constitutive relations. On the other hand, we choose to eliminate the scalar data  $\nabla_{M}(R_{s}\xi^{M})$  using the  $\varphi$  equation of motion.

### 3.4 First Derivative Corrections to Null Superfluids

Moving on to the one derivative null superfluids, let us schematically represent various quantities appearing in eqn. (3.11) up to the first order in derivatives as,

$$T^{MN} = \left[ R_n u^M u^N + 2E u^{(M} V^{N)} + P P^{MN} + R_s \xi^M \xi^N + R_v V^M V^N \right] + \mathcal{T}^{MN} + \mathcal{O}(\partial^2),$$

$$J^M = \left[ Q u^M - R_s \xi^M + Q_v V^M \right] + \mathcal{J}^M + \mathcal{O}(\partial^2),$$

$$K = \left[ -\alpha \delta_{\mathcal{B}} \varphi + \nabla_M (R_s \xi^M) \right] + \mathcal{K} + \mathcal{O}(\partial^2),$$

$$N^M = \left[ \frac{P}{T} u^M + \delta_{\mathcal{B}} \varphi R_s \xi^M + N_v V^M \right] + \mathcal{N}^M + \mathcal{O}(\partial^2),$$

$$\Delta = \alpha (\delta_{\mathcal{B}} \varphi)^2 + \mathcal{D},$$
(3.22)

where the corrections  $\mathcal{T}^{MN}$ ,  $\mathcal{J}^{M}$ ,  $\mathcal{K}$ ,  $\mathcal{N}^{M}$ ,  $\mathcal{D}$  have exactly one derivative in every term. Plugging these in the eqn. (3.11) we can get an equation among the corrections,

$$\nabla_{M} \mathcal{N}^{M} - \mathbf{N}_{\mathbf{H}}^{\perp} = \frac{1}{2} \mathcal{T}^{MN} \delta_{\mathcal{B}} g_{MN} + \mathcal{J}^{M} \delta_{\mathcal{B}} A_{M} + \mathcal{K} \delta_{\mathcal{B}} \varphi + \mathcal{D} + \mathcal{O}(\partial^{3}). \tag{3.23}$$

We will now attempt to find all the solutions to this equation, hence recovering the null superfluid constitutive relations up to the first order in derivatives.

### 3.4.1 Parity Even

We can find the most general parity even solution of eqn. (3.23) in 2 steps (note that  $N_H^{\perp}$  is parity odd): (1) first we write down the most general allowed parity-even  $\mathcal{N}^M$  and find a set of constitutive relations pertaining to that, and (2) then find the most general parity-even constitutive relations which satisfy eqn. (3.23) with  $\mathcal{N}^M = 0$ .

1. One can check that the most general form of  $\mathcal{N}^M$  (whose divergence only contains product of derivatives and has at least one  $\delta_{\mathcal{B}}$  per term) can be written as (see appendix (A) for more details),

$$\mathcal{N}^{M} = 2f_{1}u^{[M}\zeta^{N]} \frac{1}{T^{2}} \partial_{N}T + 2f_{2}u^{[M}\zeta^{N]} \partial_{N}\nu + 2f_{3}u^{[M}\zeta^{N]} \partial_{N}\nu_{n} + 2f_{4}u^{[M}\zeta^{N]} \partial_{N}R_{s} + \nabla_{N}\left(f_{5}u^{[M}\zeta^{N]}\right), \quad (3.24)$$

where f's are functions of T,  $\nu = \mu/T$ ,  $\nu_n = \mu_n/T$  and  $\hat{\mu}_s = -\frac{1}{2}\zeta^M\zeta_M$  with  $\zeta^M = P^{MN}\xi_N = \xi^M - u^M + (u^N\xi_N)V^M$  ( $P^{MN} = g^{MN} + 2u^{(M}V^N)$  is the projection operator away from the

<sup>&</sup>lt;sup>5</sup>Null and Newton-Cartan geometries behave more naturally in presence of a minimal temporal torsion  $H_{MN}=2\partial_{[M}V_{N]}$  (cref. TTNC geometries []). In presence of  $H_{MN}$ , the data  $S_2=\zeta^M\left(\frac{1}{T}\partial_MT+u^NH_{NM}\right)$  vanishes at equilibrium while  $S_{e,1}=\frac{1}{T}\zeta^M\partial_MT$  survives. However when  $H_{MN}=0$ ,  $S_2=S_{e,1}$ .

Vani	Vanishing at Equilibrium – Onshell Independent			
$S_1$	$rac{T}{2} ilde{P}^{MN}\delta_{\mathfrak{B}}g_{MN}$	$ ilde{P}^{MN} abla_{M}u_{N}$		
$S_2, S_{e,1}$ 5	$TV^{M}\zeta^{N}\delta_{\mathbb{B}}g_{MN}$	$\frac{1}{T}\zeta^{M}\nabla_{M}T$		
$S_3$	$rac{T}{2}\zeta^{M}\zeta^{N}\delta_{\mathbb{B}}g_{MN}$	$\zeta^M\zeta^N abla_M u_N$		
$S_4$	$T\zeta^{M}\delta_{\mathcal{B}}A_{M}$	$\zeta^{\scriptscriptstyle M}\left(T\nabla_{\scriptscriptstyle M}\nu+u^{\scriptscriptstyle N}F_{\scriptscriptstyle NM}\right)$		
$S_5$	$T\delta_{\mathcal{B}}arphi$	$u^M \xi_M + \mu_n - \mu$		
$V_1^{\scriptscriptstyle M}, V_{e,1}^{\scriptscriptstyle M}$	$T ilde{P}^{MR}V^N\delta_{\mathbb{B}}g_{RN}$	$\frac{1}{T}\tilde{P}^{MN}\nabla_{N}T$		
$V_2^{\scriptscriptstyle M}$	$T ilde{P}^{MR}\zeta^N\delta_{\mathfrak{B}}g_{RN}$	$2\tilde{P}^{MR}\zeta^{N} abla_{(R}u_{N)}$		
$V_3^{\scriptscriptstyle M}$	$T ilde{P}^{MN}\delta_{\mathfrak{B}}A_{N}$	$\tilde{P}^{MN}\left(T\nabla_{N}\nu+u^{R}F_{RN}\right)$		
$\sigma^{MN}$	$rac{T}{2} ilde{P}^{R\langle M} ilde{P}^{N angle S}\delta_{\mathbb{B}}g_{RS}$	$\tilde{P}^{MR}\tilde{P}^{NS}\left(\nabla_{(R}u_{S)}-rac{\tilde{P}_{RS}}{d-1}S_{1}\right)$		
$ ilde{V}_1^{\scriptscriptstyle M}$	$\epsilon^{MN}$	$V_N u_R \zeta_S V_{1,T}$		
$ ilde{V}_2^{\scriptscriptstyle M}$		$^{RST}V_Nu_R\zeta_SV_{2,T}$		
$ ilde{V}_3^{\scriptscriptstyle M}$	$\epsilon^{MN}$	$^{RST}V_Nu_R\zeta_SV_{3,T}$		
$ ilde{\sigma}^{MN}$	$\epsilon^{(M R)}$	$^{RST}V_Nu_R\zeta_SV_{3,T}$ $^{RSTP}V_Ru_S\zeta_T\sigma_P^{\ N)}$		
Van	nishing at Equilibrium	m – Onshell Dependent		
$S_6$	$Tu^{\scriptscriptstyle M}V^{\scriptscriptstyle N}\delta_{\scriptscriptstyle \mathcal{B}}g_{\scriptscriptstyle MN}$	$\frac{1}{T}u^{M}\nabla_{M}T$		
$S_7$	$Tu^{M}\delta_{\mathcal{B}}A_{M}$	$Tu^{\scriptscriptstyle M} abla_{\scriptscriptstyle M} u$		
$S_8$	$rac{T}{2}u^{M}u^{N}\delta_{\mathbb{B}}g_{MN}$	$Tu^{\scriptscriptstyle M} abla_{\scriptscriptstyle M} u_n$		
$S_9$	$Tu^{M}\zeta^{N}\delta_{\mathfrak{B}}g_{MN}$	$\zeta^{M}\left(T\nabla_{M}\nu_{n}+u^{N}\nabla_{N}u_{M}\right)$		
$V_4^{\scriptscriptstyle M}$	$T ilde{P}^{MR}u^N\delta_{\mathfrak{B}}g_{RN}$	$\tilde{P}^{MN}\left(T\nabla_{N}\nu_{n}+u^{R}\nabla_{R}u_{N}\right)$		
$ ilde{V}_4^{\scriptscriptstyle M}$	$\epsilon^{MN}$	$^{NRST}V_Nu_R\zeta_SV_{4,T}$		
	Surviving at	Equilibrium		
$S_{e,2}$		$T\zeta^{M}\partial_{M}\nu$		
$S_{e,3}$		$T\zeta^M\partial_M u_n$		
÷		:		
$V_{e,2}^{\scriptscriptstyle M}$		$T\tilde{P}^{MN}\partial_{N} u$		
$V_{e,3}^{\scriptscriptstyle M}$		$T ilde{P}^{MN}\partial_N u_n$		
:		: :		
$ ilde{ ilde{S}_{e,1}}$	$T\epsilon^{MN}$	$^{MNRST}\zeta_{M}V_{N}u_{R}\partial_{S}u_{T}$		
$ ilde{S}_{e,2}$		$rac{1}{2}T\epsilon^{MNRST}\zeta_{M}V_{N}u_{R}F_{ST}$		
:	2			
$ ilde{V}_{e,1}^{\scriptscriptstyle M}$	$T ilde{P}^M$ ,	$\frac{1}{2^{KNRST}V_Nu_R\partial_Su_T}$		
$\tilde{V}^M$		$\{\epsilon^{KNRST}V_Nu_RF_{ST}\}$		
$ ilde{V}_{e,3}^{e,2}  ilde{V}_{e,3}^{M}$	_	$e^{KNRST}\xi_N u_R \partial_S u_T$		
$\tilde{V}^M$		$rac{1}{2}T ilde{P}_{K}^{M}\epsilon^{KNRST}\xi_{N}u_{R}F_{ST}$		
• e,4 :	; ; ; ; ; ; ; ; ; ; ; ; ; ; ; ; ; ; ;			
•		•		

**Table 3:** Independent first order data for null superfluids. We have not enlisted, neither would we need, all the independent data surviving at equilibrium.

null fluid velocity). Note that,

$$\hat{\mu}_s = -\frac{1}{2} \zeta^M \zeta_M = -\frac{1}{2} \xi^M \xi_M + \xi^M u_M = \mu_s + \xi^M u_M = \mu_s - \mu_n + \mu + T \delta_{\mathcal{B}} \varphi.$$
 (3.25)

Out of the five terms in eqn. (3.24), the last one has trivially zero divergence and hence can be ignored. The forth term on the other hand can be removed by elimination of  $\nabla_M(R_s\xi^M)$  using the  $\varphi$  equation of motion. Computing the divergence of the remaining terms in  $\mathcal{N}^M$  and comparing them to eqn. (3.23), we can directly read out the corresponding null superfluid constitutive relations (the symbol ' $\ni$ ' represents that they are not yet the complete solutions of eqn. (3.23); we still have to add the terms with  $\mathcal{N}^M = 0$ ),

$$\mathcal{T}^{MN} \ni u^{M} u^{N} \left( \sum_{i=1}^{3} \alpha_{R_{n},i} S_{e,i} - \frac{1}{T} \nabla_{R} (T f_{3} \zeta^{R}) \right) + 2V^{(M} u^{N)} \left( \sum_{i=1}^{3} \alpha_{E,i} S_{e,i} - \frac{1}{T} \nabla_{R} (T f_{1} \zeta^{R}) \right)$$

$$+ \left( \zeta^{M} \zeta^{N} + 2 \zeta^{(M} u^{N)} - 2 \zeta^{(M} V^{N)} (u^{R} \xi_{R}) \right) \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i}$$

$$- 2 \xi^{(M} \sum_{i=1}^{3} f_{i} V_{e,i}^{N)} + \tilde{P}^{MN} \sum_{i=1}^{3} f_{i} S_{e,i} + 2 \zeta^{(M} V^{N)} \sum_{i=1}^{3} f_{i} S_{5+i},$$

$$\mathcal{J}^{M} \ni u^{M} \left( \sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} - \frac{1}{T} \nabla_{R} (T f_{2} \zeta^{R}) \right) - \zeta^{M} \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{3} f_{i} V_{e,i}^{M},$$

$$\mathcal{K} \ni \nabla_{M} \left( \zeta^{M} \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} - \sum_{i=1}^{3} f_{i} V_{e,i}^{M} \right),$$

$$(3.26)$$

where  $\tilde{P}^{MN} = g^{\mu\nu} + 2u^{(M}V^{N)} - \frac{1}{\zeta^R\zeta_R}\zeta^M\zeta^N$ , and we have defined,

$$df_i = \frac{\alpha_{E,i}}{T}dT + T\alpha_{R_n,i}d\nu_n + T\alpha_{Q,i}d\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s.$$
(3.27)

The actual computation is not neat and we have presented the details in appendix (B) for the aid of the readers interested in reproducing our results. Note that these constitutive relations are presented in terms of 'data' which are natural for this sector; readers can modify these to their favorite basis and get results which might look considerably messier. Moreover, these results are written in a particular 'hydrodynamic frame' chosen by aligning  $u^M$ , T,  $\mu$ ,  $\mu_n$  along  $\beta^M$ ,  $\Lambda_\beta$ , which again can be modified according to reader's preference.

2. Let us now look at the parity-even solutions to eqn. (3.23) with  $\mathcal{N}^M = 0$ ,

$$0 = \frac{1}{2} \mathcal{T}^{MN} \delta_{\mathcal{B}} g_{MN} + \mathcal{J}^{M} \delta_{\mathcal{B}} A_{M} + \mathcal{K} \delta_{\mathcal{B}} \varphi + \mathcal{D}. \tag{3.28}$$

Every term in  $\mathcal{T}^{MN}$ ,  $\mathcal{J}^{M}$ ,  $\mathcal{K}$  must either cancel or contribute to  $\Delta$  which has to be a quadratic form. It follows that the terms in  $\mathcal{T}^{MN}$ ,  $\mathcal{J}^{M}$ ,  $\mathcal{K}$  must be proportional to  $\delta_{\mathcal{B}}g_{MN}$ ,  $\delta_{\mathcal{B}}A_{M}$ ,  $\delta_{\mathcal{B}}\varphi$ . Recall however that we have chosen to eliminate  $u^{M}\delta_{\mathcal{B}}g_{MN}$ ,  $u^{M}\delta_{\mathcal{B}}A_{M}$  using the equations of motion. For  $\Delta$  to be a quadratic form, it therefore implies that  $\mathcal{T}^{MN}$ ,  $\mathcal{J}^{M}$  cannot have a term like  $\#^{(M}u^{N)}$ ,  $\#u^{M}$  respectively for some vector  $\#^{M}$  and scalar #. With this input let us write down the most generic allowed form of the currents in terms

of 34 new transport coefficients  $[\beta_{ij}]_{5\times5}$  (with  $\beta_{55}=\alpha/T$ ),  $[\kappa_{ij}]_{3\times3}$  and  $\eta$ ,

$$\mathcal{T}^{MN} \ni -T \left[ \left\{ \beta_{11} \tilde{P}^{RS} + 2\beta_{12} \zeta^{(R} V^{S)} + \beta_{13} \zeta^{R} \zeta^{S} \right\} \tilde{P}^{MN} + \left\{ \beta_{21} \tilde{P}^{RS} + 2\beta_{22} \zeta^{(R} V^{S)} + \beta_{23} \zeta^{R} \zeta^{S} \right\} 2\zeta^{(M} V^{N)} + \left\{ \beta_{31} \tilde{P}^{RS} + 2\beta_{32} \zeta^{(R} V^{S)} + \beta_{33} \zeta^{R} \zeta^{S} \right\} \zeta^{M} \zeta^{N} + 4 \left\{ \kappa_{11} V^{(R} + \kappa_{12} \zeta^{(R)} \right\} \tilde{P}^{S)(M} V^{N)} + 4 \left\{ \kappa_{21} V^{(R} + \kappa_{22} \zeta^{(R)} \right\} \tilde{P}^{S)(M} \zeta^{N)} + \eta \tilde{P}^{M\langle R} P^{S\rangle N} \right] \frac{1}{2} \delta_{\mathcal{B}} g_{RS} - T \left[ \beta_{14} \zeta^{R} \tilde{P}^{MN} + 2\beta_{24} \zeta^{R} \zeta^{(M} V^{N)} + \beta_{34} \zeta^{R} \zeta^{M} \zeta^{N} + 2\kappa_{13} \tilde{P}^{R(M} V^{N)} + 2\kappa_{23} \tilde{P}^{R(M} \zeta^{N)} \right] \delta_{\mathcal{B}} A_{R}, - T \left[ \beta_{15} \tilde{P}^{MN} + 2\beta_{25} \zeta^{(M} V^{N)} + \beta_{35} \zeta^{M} \zeta^{N} \right] \delta_{\mathcal{B}} \varphi$$

$$= -\tilde{P}^{MN} \sum_{i=1}^{5} \beta_{1i} S_{i} - 2\zeta^{(M} V^{N)} \sum_{i=1}^{5} \beta_{2i} S_{i} - \zeta^{M} \zeta^{N} \sum_{i=1}^{5} \beta_{3i} S_{i} - 2V^{(M} \sum_{i=1}^{3} \kappa_{1i} V_{i}^{N)} - 2\zeta^{(M} \sum_{i=1}^{3} \kappa_{2i} V_{i}^{N)} - \eta \sigma^{MN},$$

$$(3.29)$$

$$\mathcal{J}^{M} \ni -T \left[ \left\{ \beta_{41} \tilde{P}^{RS} + 2\beta_{42} \zeta^{(R} V^{S)} + \beta_{43} \zeta^{R} \zeta^{S} \right\} \zeta^{M} + 2 \left\{ \kappa_{31} V^{(R} + \kappa_{32} \zeta^{(R)} \right\} \tilde{P}^{S)M} \right] \frac{1}{2} \delta_{\mathcal{B}} g_{RS}$$

$$- T \left[ \beta_{44} \zeta^{M} \zeta^{N} + \kappa_{33} P^{MN} \right] \delta_{\mathcal{B}} A_{R} - T \left[ \beta_{45} \zeta^{M} \right] \delta_{\mathcal{B}} \varphi,$$

$$= -\zeta^{M} \sum_{i=1}^{5} \beta_{4i} S_{i} - \sum_{i=1}^{3} \kappa_{3i} V_{i}^{M}, \qquad (3.30)$$

$$\mathcal{K} \ni -T \left[ \beta_{51} \tilde{P}^{RS} + 2\beta_{52} \zeta^{(R} V^{S)} + \beta_{53} \zeta^{R} \zeta^{S} \right] \delta_{\mathcal{B}} g_{RS} - T \left[ \beta_{54} \zeta^{M} \right] \delta_{\mathcal{B}} A_{M} = -\sum_{i=1}^{4} \beta_{5i} S_{i}. \tag{3.31}$$

Note that we did not include a term proportional to  $\delta_{\mathcal{B}}\varphi$  in  $\mathcal{K}$ , because such a term is already present in  $K = -\alpha\delta_{\mathcal{B}}\varphi + \nabla_{M}(R_{s}\xi^{M}) + \mathcal{K} + \mathcal{O}(\partial^{2})$ . Plugging these back into eqn. (3.28) and defining  $\beta_{55} = \alpha/T$  we can read out the parity-even quadratic form  $\Delta|_{even} = \alpha(\delta_{\mathcal{B}}\varphi)^{2} + \mathcal{D}|_{even}$ ,

$$T\Delta|_{even} = \sum_{i,j=1}^{5} S_{i}\beta_{ij}S_{j} + \sum_{i,j=1}^{3} V_{i}^{M}\kappa_{ij}V_{j,M} + \eta\sigma^{MN}\sigma_{MN},$$

$$= \sum_{i,j=1}^{5} S_{i}\beta_{ij}^{(s)}S_{j} + \sum_{i,j=1}^{3} V_{i}^{M}\kappa_{ij}^{(s)}V_{j,M} + \eta\sigma^{MN}\sigma_{MN}.$$
(3.32)

In the second step we have realized that only the symmetric parts of the matrices  $\beta_{ij}$  and  $\kappa_{ij}$  will survive in this expression, and will contribute towards dissipation. Thus only 22 out of 35 transport coefficients (including  $\alpha$ ) are dissipative; the remaining 13 are non-dissipative.

### 3.4.2 Parity-Odd (5 Dimensions)

We can find the most general parity-odd solution of eqn. (3.23) in 3 steps: (1) first we consider a particular set of solutions which takes care of the anomaly  $N_H^{\perp}$  and proceed towards the non-anomalous constitutive relations, (2) then we write down the most general allowed parity-odd

 $\mathcal{N}^{M}$  and find a set of constitutive relations pertaining to that, and (2) finally find the most general parity-odd constitutive relations with zero  $\mathcal{N}^{M}$ .

1. In 4 dimensions at the first order in the derivatives  $T_{\rm H}^{M\perp}=0$  and  $J_{\rm H}^{\perp}=-\frac{3}{4}C^{(4)}\epsilon^{MNRST}u_MF_{NR}F_{SR}$  [16, 19], which implies,

$$N_{H}^{\perp} = -\frac{3}{4}C^{(4)}\frac{\mu}{T}\epsilon^{MNRST}u_{M}F_{NR}F_{SR}.$$
(3.33)

A particular solution pertaining to eqn. (3.23) with this  $N_H^{\perp}$  is given as (see [16]),

$$\mathcal{T}^{MN} \ni 6C^{(4)}\mu^2 V^{(M}B^{N)}, \quad \mathcal{J}^M \ni 6C^{(4)}\mu B^M, \quad \mathcal{K} \ni 0, \quad \mathcal{N}^M \ni 3C^{(4)}\frac{\mu^2}{T}B^M.$$
 (3.34)

Here we have defined the magnetic field and fluid vorticity as,

$$B^{M} = \frac{1}{2} \epsilon^{MNRST} V_{N} u_{R} F_{ST}, \qquad \omega^{M} = \epsilon^{MNRS\sigma} V_{N} u_{R} \partial_{S} u_{T}. \tag{3.35}$$

2. One can check that the most general form of  $\mathcal{N}^M$  (whose divergence only contains product of derivatives and has at least one  $\delta_{\mathcal{B}}$  per term) can be written as (see appendix (A) for more details),

$$\mathcal{N}^{M} = g_{1} \left( \beta^{M} \tilde{S}_{e,1} + \tilde{V}_{4}^{M} \right) + g_{2} \left( \beta^{M} \tilde{S}_{e,2} + \tilde{V}_{3}^{M} \right) + g_{3} \tilde{V}_{1}^{M} + C_{1} T \omega^{M}, \tag{3.36}$$

where g's are functions of T,  $\nu$ ,  $\hat{\mu}_s$ , and  $C_1$  is a constant<sup>6</sup>. From here we can directly read out the corresponding constitutive relations,

$$\mathcal{T}^{MN} \ni u^{M} u^{N} \sum_{i=1}^{2} \tilde{\alpha}_{R_{n},i} \tilde{S}_{e,i} + 2V^{(M} u^{N)} \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i}$$

$$+ \left( \zeta^{M} \zeta^{N} + 2\zeta^{(M} u^{N)} - 2\zeta^{(M} V^{N)} (u^{R} \xi_{R}) \right) \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \zeta^{M} \zeta^{N} \sum_{i=1}^{2} \frac{g_{i}}{2\hat{\mu}_{s}} \tilde{S}_{e,i}$$

$$- 2V^{(M} \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i+2}^{N)} - 2u^{(M} \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{N)} + 2C_{1} T^{2} V^{(M} \omega^{N)}$$

$$+ 2u^{(M} P^{N)}_{P} \epsilon^{PKRST} \nabla_{K} (Tg_{1} V_{R} u_{S} \xi_{T}) + 2V^{(M} P^{N)}_{P} \nabla_{K} (g_{3} T \epsilon^{PKRST} V_{R} u_{S} \zeta_{T}) ,$$

$$\mathcal{J}^{M} \ni u^{M} \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{e,i} - \zeta^{M} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} + \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{M} + P^{M}_{K} \epsilon^{KNRST} \nabla_{N} (Tg_{2} V_{R} u_{S} \xi_{T}) ,$$

$$\mathcal{K} \ni \nabla_{M} \left( \zeta^{M} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{M} \right) ,$$

$$(3.37)$$

where we have defined.

$$dg_i = \frac{1}{T}\tilde{\alpha}_{E,i}dT + T\tilde{\alpha}_{Q,i}d\nu + T\tilde{\alpha}_{R_n,i}d\nu_n + \left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)d\hat{\mu}_s.$$
(3.38)

The actual computation is not neat and we have presented the details in appendix (B) for interested readers.

<sup>&</sup>lt;sup>6</sup>It might be noted that since  $\nabla_M \omega^M = 0$ ,  $C_1$  a priory can be an arbitrary function rather than a constant. However, if we do the same computation in presence of torsion and later turn it off, which allows for  $\partial_{[M} V_{N]} \neq 0$ , we will be forced to set  $C_1$  to be a constant (see appendix (A) of [16]). Another way to see that  $C_1$  should be a constant is using the equilibrium partition function discussed in appendix (A).

3. We should finally consider the parity-odd constitutive relations that satisfy eqn. (3.23) with zero LHS. Following our discussion in the parity-even sector, the allowed form of the constitutive relations can be written down in terms of 10 coefficients  $[\tilde{\kappa}_{ij}]_{3\times3}$  and  $\tilde{\eta}$ ,

$$\mathcal{T}^{MN} \ni -TV_{T}u_{K}\zeta_{L} \left[ 4V^{(M}\epsilon^{N)TKL(R)} \left\{ \tilde{\kappa}_{11}V^{S)} + \tilde{\kappa}_{12}\zeta^{S)} \right\} + 4\zeta^{(M}\epsilon^{N)TKL(R)} \left\{ \tilde{\kappa}_{21}V^{S)} + \tilde{\kappa}_{22}\zeta^{S)} \right\} \right.$$

$$\left. + \tilde{\eta}\tilde{P}^{P(M}\epsilon^{N)TKL(R)}\tilde{P}^{S)}_{P} \right] \frac{1}{2}\delta_{\mathcal{B}}g_{RS} - TV_{T}u_{K}\zeta_{L} \left[ 2\tilde{\kappa}_{13}V^{(M}\epsilon^{N)TKLR} + 2\tilde{\kappa}_{23}\zeta^{(M}\epsilon^{N)TKLR} \right] \delta_{\mathcal{B}}A_{R},$$

$$= -2V^{(M)}\sum_{i=1}^{3} \tilde{\kappa}_{1i}\tilde{V}_{i}^{N)} - 2\zeta^{(M)}\sum_{i=1}^{3} \tilde{\kappa}_{2i}\tilde{V}_{i}^{N)} - \tilde{\eta}\tilde{\sigma}^{MN},$$

$$\mathcal{J}^{\mu} \ni -TV_{T}u_{K}\zeta_{L} \left[ 2\epsilon^{MTKL(R)} \left\{ \tilde{\kappa}_{31}V^{S)} + \tilde{\kappa}_{32}\zeta^{S)} \right\} \right] \frac{1}{2}\delta_{\mathcal{B}}g_{RS} - TV_{T}u_{K}\zeta_{L} \left[ \tilde{\kappa}_{33}\epsilon^{MTKLR} \right] \delta_{\mathcal{B}}A_{R},$$

$$= -\sum_{i=1}^{3} \tilde{\kappa}_{3i}\tilde{V}_{i}^{M},$$

$$\mathcal{K} \ni 0. \tag{3.39}$$

One can check that these constitutive relations trivially satisfy eqn. (3.23) with zero LHS and the quadratic form  $\Delta|_{odd} = \mathcal{D}|_{odd}$  is given as,

$$T\Delta|_{odd} \ni -\epsilon^{MNRST} V_R u_S \zeta_T \left[ \sum_{i,j=1}^3 V_{i,M} \tilde{\kappa}_{ij} V_{j,N} + \tilde{\eta} \sigma_{MP} \sigma_N^P \right],$$

$$= -\epsilon^{MNRST} V_R u_S \zeta_T \sum_{i,j=1}^3 V_{i,M} \tilde{\kappa}_{ij}^{(a)} V_{j,N}.$$
(3.40)

It follows that out of the 10 transport coefficients, only 3 contribute to dissipation and the other 7 are non-dissipative.

### 3.4.3 Positivity Constraints

The dissipative transport coefficients are required to satisfy a set of inequalities to satisfy  $\Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{even} + \mathcal{D}|_{odd} \geq 0$ ,

$$T\Delta = \sum_{i,j=1}^{5} S_i \beta_{(ij)} S_j + \left( \sum_{i,j=1}^{3} V_i^M \kappa_{(ij)} V_{j,M} + \sum_{i,j=1}^{3} V_i^M \tilde{\kappa}_{[ij]} \tilde{V}_{j,M} \right) + \eta \sigma^{MN} \sigma_{MN}.$$
 (3.41)

We want this expression to be a quadratic form, which it nearly is except the parity-odd terms in the brackets. However this term can be made into a quadratic form by noticing that the square of a parity odd term is parity-even, due to the identity,

$$(\epsilon^{MNRST} V_R u_S \zeta_T) (\epsilon_{MKLOP} V^L u^O \zeta^P) = \tilde{P}_K^N \zeta^M \zeta_M = -2\hat{\mu}_s \tilde{P}_K^N, \tag{3.42}$$

We define,

$$\begin{pmatrix} V_1'^M \\ V_2'^M \\ V_3'^M \end{pmatrix} = \begin{pmatrix} V_1^M \\ V_2^M \\ V_3^M \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}_1'^M \\ \tilde{V}_2'^M \\ \tilde{V}_3'^M \end{pmatrix}, \qquad \kappa'_{ij} = \kappa_{ij} + k_{ij}, \qquad k_{[ij]} = 0, \qquad (3.43)$$

such that,

$$\sum_{i,j=1}^{3} V_i^{\prime M} \kappa_{(ij)}^{\prime} V_{j,M}^{\prime} = \sum_{i,j=1}^{3} V_i^{M} \kappa_{(ij)} V_{j,M} + \sum_{i,j=1}^{3} V_i^{M} \tilde{\kappa}_{[ij]} \tilde{V}_{j,M}.$$
(3.44)

Using the identity eqn. (3.42), the above equation can be easily solved to give,

$$[a_{ij}] = \begin{pmatrix} 0 & \frac{\tilde{\kappa}_{[12]}}{\kappa_{11}} & \frac{\kappa_{11} \left(\kappa_{22}\tilde{\kappa}_{[13]} - \kappa_{(12)}\tilde{\kappa}_{[23]}\right) - \tilde{\kappa}_{[12]} \left(\kappa_{(12)}\kappa_{(13)} + \zeta^{M}\zeta_{M}\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}\right)}{\kappa_{11} \left(\kappa_{11}\kappa_{22} - \kappa_{(12)}^{2} - \zeta^{M}\zeta_{M}\tilde{\kappa}_{[12]}^{2}\right)} \\ 0 & 0 & \frac{\kappa_{11}\tilde{\kappa}_{[23]} - \kappa_{(12)}\tilde{\kappa}_{[13]} + \kappa_{(13)}\tilde{\kappa}_{[12]}}{\kappa_{11}\kappa_{22} - \kappa_{(12)}^{2} - \zeta^{M}\zeta_{M}\tilde{\kappa}_{[12]}^{2}} \end{pmatrix},$$
(3.45)

$$[k_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\zeta^M \zeta_M \frac{(\tilde{\kappa}_{[12]})^2}{\kappa_{11}} & -\zeta^M \zeta_M \frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} \\ 0 & -\zeta^M \zeta_M \frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} & -\zeta^M \zeta_M \left( \frac{(\tilde{\kappa}_{[13]})^2}{\kappa_{11}} + \frac{(\kappa_{11}\tilde{\kappa}_{[23]} - \kappa_{(12)}\tilde{\kappa}_{[13]} + \kappa_{(13)}\tilde{\kappa}_{[12]})^2}{\kappa_{11}(\kappa_{11}\kappa_{22} - \kappa_{(12)}^2 - \zeta^M \zeta_M \tilde{\kappa}_{[12]}^2)} \right) \end{pmatrix}.$$
(3.46)

Consequently  $\Delta$  will take the form,

$$T\Delta = \sum_{i,j=1}^{5} S_i \beta_{(ij)} S_j + \sum_{i,j=1}^{3} V_i^{\prime M} \kappa_{(ij)}^{\prime} V_{j,M}^{\prime} + \eta \sigma^{MN} \sigma_{MN}.$$
 (3.47)

Given  $T \geq 0$ , the condition  $\Delta \geq 0$  implies that  $\eta \geq 0$  and the matrices  $[\beta_{(ij)}]_{5\times 5}$ ,  $[\kappa'_{(ij)}]_{3\times 3}$  have all non-negative eigenvalues. This gives 9 inequalities among 25 dissipative transport coefficients, and 16 are completely arbitrary.

### 3.5 Summary

We have completed the analysis of a null superfluid up to the first order in derivatives. Here we summarize the results. We found that the entire null superfluid transport up to the first order in derivatives is characterized by an ideal order pressure P, 51 first order transport coefficients which are functions of T,  $\mu/T$ ,  $\mu_n/T$ ,  $\hat{\mu}_s$ , and two constants  $C_1$ ,  $C^{(4)}$ . P,  $C_1$  and  $C^{(4)}$  along with 6 transport coefficients,

Parity Even (3): 
$$f_1$$
,  $f_2$ ,  $f_3$ , Parity Odd (3):  $g_1$ ,  $g_2$ ,  $g_3$ , (3.48)

totally determine the *hydrostatic* transport (part of the constitutive relations that survive at equilibrium). *Non-hydrostatic non-dissipative* transport (part that does not survive at equilibrium but doesn't contribute to  $\Delta \geq 0$  either) is given by 20 transport coefficients,

Parity Even (13): 
$$[\beta_{[ij]}]_{5\times 5}$$
 (antisymmetric),  $[\kappa_{[ij]}]_{3\times 3}$  (antisymmetric),  
Parity Odd (7):  $[\tilde{\kappa}_{(ij)}]_{3\times 3}$  (symmetric),  $\tilde{\eta}$ . (3.49)

Finally the entire dissipative transport is given by 25 transport coefficients,

Parity Even (22): 
$$[\beta_{(ij)}]_{5\times5}$$
 (symmetric),  $[\kappa_{(ij)}]_{3\times3}$  (symmetric),  $\eta$ ,

Parity Odd (3):  $[\tilde{\kappa}_{[ij]}]_{3\times3}$  (antisymmetric). (3.50)

These dissipative transport coefficients follow a set of inequalities  $(\kappa'_{ij})$  is defined in eqn. (3.43),

$$[\beta_{(ij)}]_{5\times 5}, \quad [\kappa'_{(ij)}]_{3\times 3}, \quad \eta \geq 0,$$
 (3.51)

where a 'non-negative matrix' implies all its eigenvalues are non-negative. Using P,  $f_i$ ,  $g_i$  we define some new functions,

$$\mathrm{d}P = S\mathrm{d}T + Q\mathrm{d}\mu + R_n\mathrm{d}\mu_n + R_s\mathrm{d}\mu_s, \qquad E + P = ST + Q\mu + R_n\mu_n,$$

$$\mathrm{d}f_i = \frac{\alpha_{E,i}}{T}\mathrm{d}T + T\alpha_{R_n,i}\mathrm{d}\nu_n + T\alpha_{Q,i}\mathrm{d}\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s, \quad \alpha_{E,i} + f_i = \alpha_{S,i}T + \alpha_{Q,i}\mu + \alpha_{R_n,i}\mu_n,$$

$$\mathrm{d}g_i = \frac{\tilde{\alpha}_{E,i}}{T}\mathrm{d}T + T\tilde{\alpha}_{R_n,i}\mathrm{d}\nu_n + T\tilde{\alpha}_{Q,i}\mathrm{d}\nu + \left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s, \quad \tilde{\alpha}_{E,i} + g_i = \tilde{\alpha}_{S,i}T + \tilde{\alpha}_{Q,i}\mu + \tilde{\alpha}_{R_n,i}\mu_n.$$

$$(3.52)$$

In terms of these transport coefficients, corrections to the Josephson equation (K = 0) coming from the first order null superfluid transport are given as (here  $\beta_{55} = \alpha/T$ ),

$$u^{M}\xi_{M} + \mu_{n} - \mu = \frac{1}{\beta_{55}} \nabla_{M}(R_{s}\xi^{M}) - \sum_{i=1}^{4} \frac{\beta_{5i}}{\beta_{55}} S_{i}$$

$$+ \frac{1}{\beta_{55}} \nabla_{M} \left( \zeta^{M} \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \zeta^{M} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \sum_{i=1}^{3} f_{i} V_{e,i}^{M} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{M} \right) + \mathcal{O}(\partial^{2}), \quad (3.53)$$

which can be seen as determining  $u^M \xi_M$  in terms of the other null superfluid variables. Note that though this equation contains second order terms, it is only correct up to the first order in derivatives, and will admit further corrections coming from higher order null superfluid transport. The energy-momentum tensor, charge current and entropy current up to first order in derivatives are however given as,

$$T^{MN} = R_n u^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + R_s \xi^M \xi^N + \mathcal{T}^{MN} + \mathcal{O}(\partial^2),$$
  

$$J^M = Qu^M - R_s \xi^M + \mathcal{J}^M + \mathcal{O}(\partial^2)$$
  

$$J_S^M = Su^M + \mathcal{S}^M + \mathcal{O}(\partial^2),$$
(3.54)

where the higher derivative corrections are,

$$\begin{split} \mathcal{T}^{MN} &= u^{M} u^{N} \left[ \sum_{i=1}^{3} \alpha_{R_{n,i}} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{n,i}} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{R} (T f_{3} \zeta^{R}) \right] \\ &+ 2 V^{(M} u^{N)} \left[ \sum_{i=1}^{3} \alpha_{E,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{R} (T f_{1} \zeta^{R}) \right] \\ &+ 2 \zeta^{(M} u^{N)} \left[ \sum_{i=1}^{3} \alpha_{R_{s,i}} S_{c,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} \right] \\ &+ 2 \zeta^{(M} V^{N)} \left[ \sum_{i=1}^{3} f_{i} S_{5+i} - (u^{N} \xi_{R}) \left( \sum_{i=1}^{3} \tilde{\alpha}_{R_{s,i}} S_{c,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} \right) \right] \\ &+ 2 \zeta^{(M} V^{N)} \left[ \sum_{i=1}^{3} f_{i} S_{5+i} - (u^{N} \xi_{R}) \left( \sum_{i=1}^{3} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{c,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} - \sum_{i=1}^{5} \tilde{\beta}_{3i} S_{i} \right) \right] \\ &+ 2 \zeta^{(M} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{N} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{N} + P^{N}_{p} e^{PKRST} \nabla_{K} \left( T g_{1} V_{R} u_{S} \xi_{T} \right) \right] \\ &+ 2 u^{(M)} \left[ \left( u^{R} \xi_{R} \right) \sum_{i=1}^{3} f_{i} V_{e,i}^{N} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{N} + P^{N}_{p} e^{PKRST} \nabla_{K} \left( T g_{1} V_{R} u_{S} \xi_{T} \right) \right] \\ &+ 2 V^{(M)} \left[ \left( u^{R} \xi_{R} \right) \sum_{i=1}^{3} f_{i} V_{e,i}^{N} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{N} - \sum_{i=1}^{3} \kappa_{1i} V_{i}^{N} - \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}_{i}^{N} + 3C^{(4)} \mu^{2} B^{N} \right] \\ &+ P^{N}_{p} e^{PKRST} \nabla_{K} \left( T g_{3} V_{R} u_{S} \xi_{T} \right) + C_{1} T^{2} \omega^{N} \right] + \tilde{P}^{MN} \left[ \sum_{i=1}^{3} f_{i} S_{e,i} - \sum_{i=1}^{5} \beta_{1i} S_{i} \right] \\ &- 2 \zeta^{(M)} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{N} + \sum_{i=1}^{3} \kappa_{2i} V_{i}^{N} + \sum_{i=1}^{3} \tilde{\kappa}_{2i} \tilde{V}_{i}^{N} \right] + p^{M}_{K} e^{KNRST} \nabla_{N} \left( T g_{2} V_{R} u_{S} \xi_{T} \right) \\ &- \zeta^{M} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{N} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{q,i} - \frac{1}{T} \nabla_{R} \left( T f_{2} \zeta^{N} \right) \right] + P^{M}_{K} e^{KNRST} \nabla_{N} \left( T g_{2} V_{R} u_{S} \xi_{T} \right) \\ &- \zeta^{M} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{N} + \sum_{i=1}^{3} \tilde{\alpha}_{3i} V_{i}^{M} - \sum_{i=1}^{3} \tilde{\kappa}_{3i} \tilde{V}_{i}^{N} + 6C^{(4)} \mu B^{M}, \right. \quad (3.56) \\ \mathcal{S}^{M} = u^{M} \left[ \sum_{i=1}^{3} \alpha_{S,i} \tilde{S}_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^{2}} \nabla_{R} \left( T f_{1} \zeta^{N} \right) + \frac{\mu_{n}}{T^{2}} \nabla_{R} \left( T f_{3} \zeta^{R} \right) + \frac{\mu_{n}}{T^$$

The scalar  $S_5 = T\delta_{\mathcal{B}}\varphi = u^M\xi_M + \mu_n - \mu$  appearing here can be eliminated in favor of  $\nabla_M(R_s\xi^M)$  using the Josephson equation. We will like to reiterate that these results are presented in a particular hydrodynamic frame (gained by aligning  $u^M$ , T,  $\mu_n$ ,  $\mu$  along  $\beta^{\mu}$ ,  $\Lambda_{\beta}$ ) and in a "natural" choice of basis for the independent data. They can be transformed to any other preferred hydrodynamic frame or basis by a straight forward substitution.

## 4 | Null Reduction to Galilean Superfluids

We now reduce our null superfluid results to Galilean superfluids. The results are presented in the covariant Newton-Cartan notation and the conventional non-covariant notation (for superfluids coupled to flat space-time). For more details on the reduction, please refer [16].

### 4.1 Newton-Cartan Notation

We start with a quick review of null reduction of null backgrounds to Newton-Cartan backgrounds; for details see [16]. For an excellent review of Newton-Cartan geometries, please refer the appendix of [32].

Background and Hydrodynamic Fields: On our null background  $\mathcal{M}_{(d+1)}$ , we choose a basis  $\{x^M\} = \{x^-, x^\mu\}$  such that the null isometry  $\mathcal{V} = \{V = \partial_-, \Lambda_V = 0\}$ . The fact that  $\mathcal{V}$  is an isometry implies that all the fields in the theory are independent of the  $x^-$  coordinate. To perform the reduction, we require an arbitrary null field  $v^M$  normalized as  $v^M v_M = 0$ ,  $v^M V_M = -1$ , which can be interpreted as providing a "Galilean frame of reference". In the case of a null (super)fluid, the null fluid velocity  $v^M = u^M$  defines a special Galilean frame which we refer to as the "fluid frame of reference". In an arbitrary Galilean frame, we decompose the fields  $V^M$ ,  $v^M$ ,  $g_{MN}$ ,  $A_M$  in the chosen basis as,

$$V^{M} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^{M} = \begin{pmatrix} v^{\mu} B_{\mu}^{(v)} \\ v^{\mu} \end{pmatrix}, \quad g_{MN} = \begin{pmatrix} 0 & -n_{\nu} \\ -n_{\mu} & h_{\mu\nu} + 2n_{(\mu} B_{\nu)}^{(v)} \end{pmatrix}, \quad A_{M} = \begin{pmatrix} -1 \\ A_{\mu} \end{pmatrix}, \quad (4.1)$$

along with,

$$V_{M} = \begin{pmatrix} 0 \\ -n_{\mu} \end{pmatrix}, \quad v_{M} = \begin{pmatrix} -1 \\ B_{\mu}^{(v)} \end{pmatrix}, \quad g^{MN} = \begin{pmatrix} h^{\nu\rho}B_{\nu}^{(v)}B_{\rho}^{(v)} - 2v^{\mu}B_{\mu}^{(v)} & h^{\nu\rho}B_{\rho}^{(v)} - v^{\nu} \\ h^{\mu\nu}B_{\nu}^{(v)} - v^{\mu} & h^{\mu\nu} \end{pmatrix}, \quad (4.2)$$

such that,

$$n_{\mu}v^{\mu} = 1, \qquad v^{\mu}h_{\mu\nu} = 0, \qquad n_{\mu}h^{\mu\nu} = 0, \qquad h_{\mu\rho}h^{\rho\nu} + n_{\mu}v^{\nu} = \delta_{\mu}^{\ \nu}.$$
 (4.3)

The collection of fields  $\{n_{\mu}, v^{\mu}, h^{\mu\nu}, h_{\mu\nu}, B^{(v)}_{\mu}\}$  defines a Newton-Cartan structure. The condition  $\nabla_{M}V^{N}=0$  implies that the "time-metric"  $n=n_{\mu}\mathrm{d}x^{\mu}$  is a closed one-form, i.e.  $\mathrm{d}n=0$ ; this is known to be true for torsionless Newton-Cartan structures. Note that after choosing the said basis, the residual diffeomorphisms are  $x^{\mu}\to x^{\mu}+\chi^{\mu}(x^{\nu})$  and  $x^{-}\to \xi^{-}+\chi^{-}(x^{\mu})$ . The former of these are just the Newton-Cartan diffeomorphisms, while the latter are known as "mass gauge transformations". Only fields that transform under these mass gauge transformations are,

$$\delta_{\chi^{-}}B_{\mu}^{(v)} = -\partial_{\mu}\chi^{-}, \qquad \delta_{\chi^{-}}A_{\mu} = -\partial_{\mu}\chi^{-}. \tag{4.4}$$

 $B_{\mu}^{(v)}$  is therefore known as the mass gauge field. On the other hand mass gauge transformation of  $A_{\mu}$  can be absorbed into its U(1) gauge transformation. We define the volume element on a Newton-Cartan background as,

$$\varepsilon^{\mu\nu\rho\sigma} = v_M \epsilon^{M\mu\nu\rho\sigma} = -\epsilon^{-\mu\nu\rho\sigma}. \tag{4.5}$$

Note that the volume element is independent of the Galilean frame employed to define it. The Levi-Civita connection  $\Gamma^R_{MN}$  decomposes in this basis as,

$$\Gamma^{\lambda}_{\mu\nu} = v^{\lambda}\partial_{(\mu}n_{\nu)} + \frac{1}{2}h^{\lambda\rho}\left(\partial_{\mu}h_{\rho\nu} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}\right) - \Omega^{(v)}_{\sigma(\mu}n_{\nu)}h^{\sigma\lambda},$$

$$\Gamma^{-}_{\mu\nu} = h_{\lambda(\mu}\nabla_{\nu)}v^{\lambda} - \nabla_{(\mu}B^{(v)}_{\nu)},$$
(4.6)

and all the remaining components zero. Here we have identified  $\Gamma^{\lambda}_{\mu\nu}$  as the (torsionless) Newton-Cartan connection and denoted the respective covariant derivative by  $\nabla_{\mu}$ . We have also defined the (dual) frame vorticity and electromagnetic field strength as,

$$\Omega_{\mu\nu}^{(v)} = 2h_{\sigma[\nu}\nabla_{\mu]}v^{\sigma} = \partial_{\mu}B_{\nu}^{(v)} - \partial_{\nu}B_{\mu}^{(v)}, \qquad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{4.7}$$

The covariant derivative  $\nabla$  acts on the Newton-Cartan structure appropriately,

$$\nabla_{\mu} n_{\nu} = 0, \qquad \nabla_{\mu} h^{\rho\sigma} = 0, \qquad \nabla_{\mu} h_{\nu\rho} = -2n_{(\nu} h_{\rho)\sigma} \nabla_{\mu} v^{\sigma}. \tag{4.8}$$

Note that  $v^M$  was an arbitrary field chosen to perform the reduction, and one is allowed to arbitrarily redefine it without changing the physics. This leads to the invariance of the system under "Milne transformations" of the Newton-Cartan structure,

$$v^{\mu} \to v^{\mu} + \psi^{\mu}, \quad h_{\mu\nu} \to h_{\mu\nu} - 2n_{(\mu}\psi_{\nu)} + n_{\mu}n_{\nu}\psi^{\rho}\psi_{\rho}, \quad B_{\mu}^{(v)} \to B_{\mu}^{(v)} + \psi_{\mu} - \frac{1}{2}n_{\mu}\psi^{\rho}\psi_{\rho}, \quad (4.9)$$

where  $\psi^{\mu}n_{\mu} = 0$ ,  $\psi_{\mu} = h_{\mu\nu}\psi^{\nu}$ . The fields  $n_{\mu}$ ,  $h^{\mu\nu}$ ,  $\Gamma^{\rho}_{\mu\nu}$  and  $\varepsilon^{\mu\nu\rho\sigma}$  are Milne invariant. We can now decompose the fluid velocity  $u^{M}$  and the associated projector  $P^{MN}$  as,

$$u^{M} = \begin{pmatrix} u^{\mu} B_{\mu} \\ u^{\mu} \end{pmatrix}, \quad u_{M} = \begin{pmatrix} -1 \\ B_{\mu} \end{pmatrix}, \quad P_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & p_{\mu\nu} \end{pmatrix}, \quad P^{MN} = \begin{pmatrix} p^{\nu\rho} B_{\nu} B_{\rho} & p^{\mu\nu} B_{\nu} \\ p^{\mu\nu} B_{\nu} & p^{\mu\nu} \end{pmatrix}. \quad (4.10)$$

The fields  $\{n_{\mu}, u^{\mu}, p^{\mu\nu}, p_{\mu\nu}, B_{\mu}\}$  define the Newton-Cartan structure in the fluid frame of reference, satisfying,

$$n_{\mu}u^{\mu} = 1,$$
  $u^{\mu}p_{\mu\nu} = 0,$   $n_{\mu}p^{\mu\nu} = 0,$   $p_{\mu\rho}p^{\rho\nu} + n_{\mu}u^{\nu} = \delta_{\mu}^{\ \nu}.$  (4.11)

They can be re-expressed in terms of  $\{n_{\mu}, v^{\mu}, h^{\mu\nu}, h_{\mu\nu}, B^{(v)}_{\mu}\}$  using eqn. (4.9) with  $\psi^{\mu} = \bar{u}^{\mu} = h^{\mu}_{\ \nu} u^{\nu} = u^{\mu} - v^{\mu}$ ,

$$p^{\mu\nu} = h^{\mu\nu}, \quad p_{\mu\nu} = h_{\mu\nu} - 2n_{(\mu}\bar{u}_{\nu)} + n_{\mu}n_{\nu}\bar{u}^{\rho}\bar{u}_{\rho}, \quad B_{\mu} = B_{\mu}^{(v)} + \bar{u}_{\mu} - \frac{1}{2}n_{\mu}\bar{u}^{\rho}\bar{u}_{\rho}. \tag{4.12}$$

The (dual) fluid vorticity is defined similar to the (dual) frame vorticity as,

$$\Omega_{\mu\nu} = 2p_{\sigma[\nu} \nabla_{\mu]} u^{\sigma} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}. \tag{4.13}$$

For later use, we define the magnetic field and fluid vorticity,

$$B^{\mu} = \frac{1}{2} \varepsilon^{\nu\rho\sigma\mu} n_{\nu} F_{\rho\sigma}, \qquad \omega^{\mu} = \frac{1}{2} \varepsilon^{\nu\rho\sigma\mu} n_{\nu} \Omega_{\rho\sigma}. \tag{4.14}$$

Finally the superfluid velocity can be decomposed as

$$\zeta^{M} = \begin{pmatrix} B_{\mu} \zeta^{\mu} \\ \zeta^{\mu} \end{pmatrix}, \qquad \xi^{M} = \begin{pmatrix} \mu_{s} + \frac{1}{2} p_{\mu\nu} \zeta^{\mu} \zeta^{\nu} + B_{\mu} \xi^{\mu} \\ \xi^{\mu} = \zeta^{\mu} + u^{\mu} \end{pmatrix}, \tag{4.15}$$

where  $\xi^{\mu}n_{\mu} = 1$ ,  $\zeta^{\mu}n_{\mu} = 0$ . We have treated the superfluid potential  $\mu_s$  as an independent component of  $\xi^M$ . The hatted superfluid potential is however given as  $\hat{\mu}_s = -\frac{1}{2}\zeta^{\mu}\zeta_{\mu}$ . Decomposition of the projector  $\tilde{P}^{MN}$  on the other hand is,

$$\tilde{P}_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\mu\nu} = p_{\mu\nu} - \frac{\zeta_{\mu}\zeta_{\nu}}{p^{\rho\sigma}\zeta_{\rho}\zeta_{\sigma}} \end{pmatrix}, \quad \tilde{P}^{MN} = \begin{pmatrix} \tilde{p}^{\nu\rho}B_{\nu}B_{\rho} & \tilde{p}^{\mu\nu}B_{\nu} \\ \tilde{p}^{\mu\nu}B_{\nu} & \tilde{p}^{\mu\nu} = p^{\mu\nu} - \frac{\zeta^{\mu}\zeta^{\nu}}{p^{\rho\sigma}\zeta_{\rho}\zeta_{\sigma}} \end{pmatrix}. \tag{4.16}$$

Currents and Conservation: The mass current  $\rho^{\mu}$ , energy current  $\epsilon^{\mu}$ , stress tensor  $t^{\mu\nu}$ , charge current  $j^{\mu}$  and entropy current  $s^{\mu}$  on Newton-Cartan backgrounds can be respectively read out in terms of  $T^{MN}$ ,  $J^{M}$ ,  $J^{M}_{S}$  as [16],

$$\rho^{\mu} = -T^{\mu M} V_M, \qquad \epsilon^{\mu} = -T^{\mu M} u_M, \qquad t^{\mu \nu} = P^{\mu}_{\ M} P^{\nu}_{\ N} T^{MN}, \qquad j^{\mu} = J^{\mu}, \qquad s^{\mu} = J^{\mu}_{\ S}, \quad (4.17)$$

with  $t^{\mu\nu} = t^{\nu\mu}$  and  $t^{\mu\nu}n_{\nu} = 0$ . They satisfy the conservation laws and the second law of thermodynamics,

Mass Conservation:  $\nabla_{\mu}\rho^{\mu} = 0$ ,

Energy Conservation:  $\nabla_{\mu} \epsilon^{\mu} = -u^{\nu} F_{\nu\rho} j^{\rho} - (u^{\mu} \rho^{\sigma} + t^{\mu\sigma}) p_{\sigma\nu} \nabla_{\mu} u^{\nu} - T_{H\nu}^{\perp} u^{\nu},$ 

 $\text{Momentum Conservation:} \ \nabla_{\mu}(u^{\mu}p^{\sigma}_{\ \nu}\rho^{\nu}+t^{\mu\sigma})=p^{\sigma\nu}F_{\nu\rho}j^{\rho}-\rho^{\mu}\nabla_{\mu}u^{\sigma}+\mathrm{T_{H}}^{\perp}_{\ \nu}p^{\sigma\nu},$ 

Charge Conservation:  $\nabla_{\mu} j^{\mu} = J_{H}^{\perp}$ ,

Second Law of Thermo.:  $\nabla_{\mu} s^{\mu} \ge 0.$  (4.18)

The energy current  $\epsilon^{\mu}$  and the stress tensor  $t^{\mu\nu}$  in eqn. (4.17) are defined in the fluid frame of reference; we can define the respective quantities in an arbitrary frame of reference,

$$\epsilon_{(v)}^{\mu} = -T^{\mu M} v_{M} = \epsilon^{\mu} + u^{\mu} \bar{u}^{\nu} p_{\nu \rho} \rho^{\rho} + \frac{1}{2} \rho^{\mu} \bar{u}^{\rho} \bar{u}_{\rho} + t^{\mu \nu} \bar{u}_{\nu}, 
t_{(v)}^{\mu \nu} = (P_{(v)})^{\mu}_{M} (P_{(v)})^{\nu}_{N} T^{MN} = t^{\mu \nu} + 2 \bar{u}^{(\mu} h^{\nu)}_{\sigma} \rho^{\sigma} - \bar{u}^{\mu} \bar{u}^{\nu} \rho^{\sigma} n_{\sigma},$$
(4.19)

where  $P_{(v)}^{MN} = g^{MN} + 2v^{(M}V^{N)}$ . They satisfy the conservation laws,

$$\nabla_{\mu} \epsilon^{\mu}_{(v)} = -v^{\nu} F_{\nu \rho} j^{\rho} - \left( v^{\mu} \rho^{\sigma} + t^{\mu \sigma}_{(v)} \right) h_{\sigma \nu} \nabla_{\mu} v^{\nu} - T_{H}^{\perp}_{\nu} v^{\nu}$$

$$\nabla_{\mu} (v^{\mu} h^{\sigma}_{\nu} \rho^{\nu} + t^{\mu \sigma}_{(v)}) = h^{\sigma \nu} F_{\nu \rho} j^{\rho} - \rho^{\mu} \nabla_{\mu} v^{\sigma} + T_{H}^{\perp}_{\nu} h^{\sigma \nu}. \tag{4.20}$$

Galilean Superfluid Constitutive Relations: Finally, by a direct computation we can find that the Galilean superfluid constitutive relations in the fluid frame take a structural form.

$$\rho^{\mu} = \rho u^{\mu} + R_{s} \xi^{\mu} + \varsigma^{\mu}_{\rho}, 
\epsilon^{\mu} = \epsilon u^{\mu} + R_{s} \left(\frac{1}{2} \zeta^{\mu} \zeta_{\mu} + \mu_{s}\right) \xi^{\mu} + \varsigma^{\mu}_{\epsilon}, 
t^{\mu\nu} = P p^{\mu\nu} + R_{s} \zeta^{\mu} \zeta^{\nu} + \varsigma^{\mu\nu}_{t}, 
j^{\mu} = q u^{\mu} - R_{s} \xi^{\mu} + \varsigma^{\mu}_{q}, 
s^{\mu} = s u^{\mu} + \varsigma^{\mu}_{s}.$$
(4.21)

While in an arbitrary frame, energy current and stress tensor are given as,

$$\epsilon_{(v)}^{\mu} = u^{\mu} \left( \epsilon + \frac{1}{2} \rho \bar{u}^{2} + \varsigma_{\rho}^{\sigma} \bar{u}_{\sigma} \right) + R_{s} \xi^{\mu} \left( \frac{1}{2} \bar{\xi}^{2} + \mu_{s} \right) + P \bar{u}^{\mu} + \left( \varsigma_{\epsilon}^{\mu} + \frac{1}{2} \varsigma_{\rho}^{\mu} \bar{u}^{2} + \varsigma_{s}^{\mu\rho} \bar{u}_{\rho} \right),$$

$$t_{(v)}^{\mu\nu} = \rho \bar{u}^{\mu} \bar{u}^{\nu} + R_{s} \bar{\xi}^{\mu} \bar{\xi}^{\nu} + P h^{\mu\nu} + \left( \varsigma_{s}^{\mu\nu} + 2 \varsigma_{\rho}^{(\mu} \bar{u}^{\nu)} \right),$$
(4.22)

	Newton-Cartan Data	Non-Covariant Data		
	Vanishing at Equilibrium –	Onshell Independent		
$S_1$	$\tilde{p}^{\mu}_{ u} abla_{\mu}u^{ u}$	$S_1$	$ ilde{p}^{ij}\partial_i u_j$	
$S_2, S_{e,1}$	$rac{1}{T}\zeta^{\mu}\partial_{\mu}T$	$S_2, S_{e,1}$	$\frac{1}{T}\zeta^i\partial_i T$	
$S_3$	$\zeta^{\mu}\zeta_{ u} abla_{\mu}u^{ u}$	$S_3$	$\zeta^i\zeta_j\partial_i u^j$	
$S_4$	$\zeta^{\mu} \left( T \partial_{\mu} \nu + u^{\nu} F_{\nu \mu} \right)$	$S_4$	$\zeta^i(T\partial_i\nu - e_i + u^j\beta_{ji})$	
$S_5$	$-\frac{1}{2}\zeta^{\mu}\zeta_{\mu} - \mu_s + \mu_n - \mu$	$S_5$	$-\frac{1}{2}\zeta^k\zeta_k - \mu_s + \mu_n - \mu$ $\frac{1}{T}\tilde{p}^{ij}\partial_j T$	
$V_1^\mu, V_{e,1}^\mu$	$rac{1}{T} ilde{p}^{\mu u}\partial_{ u}T$	$V_1^i, V_{e,1}^i$		
$V_2^{\mu}$	$2\tilde{p}^{\mu\nu}\zeta^{\sigma}p_{\rho(\sigma}\nabla_{\nu)}u^{\rho}$	$V_2^i$	$\tilde{p}^{ij}\zeta^k\partial_{(j}u_{k)}$	
$V_3^{\mu}$	$\tilde{p}^{\mu\nu} \left( T \partial_{\nu} \nu + u^{\rho} F_{\rho\nu} \right)$ $\tilde{p}^{\mu\rho} \tilde{p}^{\nu\sigma} \left( p_{\tau(\rho} \nabla_{\sigma)} u^{\tau} - \frac{\tilde{p}_{\rho\sigma}}{d-1} S_1 \right)$	$V_3^i$	$\tilde{p}^{ij}(T\partial_j\nu - e_j + u^k\beta_{kj})$	
$\sigma^{\mu  u}$	$\tilde{p}^{\mu\rho}\tilde{p}^{\nu\sigma}\left(p_{\tau(\rho}\nabla_{\sigma)}u^{\tau}-\frac{\tilde{p}_{\rho\sigma}}{d-1}S_{1}\right)$	$\sigma^{ij}$	$\tilde{p}^{ik}\tilde{p}^{jl}(\partial_{(k}u_{l)}+\frac{\tilde{p}_{kl}}{d-1}S_{1})$	
$ ilde{V}_1^\mu$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{1,\sigma}$	$ ilde{V}_1^i$	$arepsilon^{ijk}\zeta_j V_{1,k}$	
$ ilde{V}_2^\mu$	$-arepsilon^{\mu u ho\sigma}n_ u\zeta_ ho V_{2,\sigma}$	$ ilde{V}_2^i$	$arepsilon^{ijk}\zeta_j V_{2,k}$	
$ ilde{V}_3^\mu$	$-\varepsilon^{\mu u ho\sigma}n_{ u}\zeta_{ ho}V_{3,\sigma}$	$egin{array}{c}  ilde{V}_2^i \  ilde{V}_3^i \  ilde{\sigma}^{ij} \end{array}$	$\frac{\varepsilon^{ijk}\zeta_{j}V_{3,k}}{\varepsilon^{(i kl}\zeta_{k}\sigma_{l}^{\ j)}}$	
$\tilde{\sigma}^{\mu u}$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{3,\sigma} \\ -\varepsilon^{(\mu \rho\sigma\tau}n_{\rho}\zeta_{\sigma}\sigma_{\tau}^{\nu)}$	$ ilde{\sigma}^{ij}$	$arepsilon^{(i kl}\zeta_k\sigma_l^{\ j)}$	
	Vanishing at Equilibrium -	- Onshell I	Dependent	
$S_6$	$\frac{1}{T}u^{\mu}\partial_{\mu}T$	$S_6$	$\frac{1}{T}(\partial_t T + u^i \partial_i T)$	
$S_7$	$T u^\mu \partial_\mu  u$	$S_7$	$T(\partial_t \nu + u^i \partial_i \nu)$	
$S_8$	$Tu^{\mu}\partial_{\mu} u_{n}$	$S_8$	$T(\partial_t \nu_n + u^i \partial_i \nu_n)$	
$S_9$	$\zeta^{\mu} \left( T \partial_{\mu} \nu_n + u^{\nu} p_{\rho\mu} \nabla_{\nu} u^{\rho} \right)$	$S_9$	$\zeta^i(T\partial_i\nu_n + \partial_t u_i + u^j\partial_j u_i)$	
$V_4^{\mu}$	$\tilde{P}^{\mu\nu} \left( T \partial_{\nu} \nu_n + u^{\sigma} p_{\rho\nu} \nabla_{\sigma} u^{\rho} \right)$	$V_4^i$	$\tilde{p}^{ij}(T\partial_j\nu_n + \partial_t u_j + u^k \partial_k u_j)$	
$ ilde{V}_4^\mu$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{4,\sigma}$	$ ilde{V}_4^i$	$ \varepsilon^{ijk}\zeta_j V_{4,k} $	
	Surviving at Eq	uilibrium		
$S_{e,2}$	$T\zeta^{\mu}\partial_{\mu} u$	$S_{e,2}$	$T\zeta^i\partial_i u$	
$S_{e,3}$	$T\zeta^{\mu}\partial_{\mu} u_{n}$	$S_{e,3}$	$T\zeta^i\partial_i u_n$	
÷	÷	÷	i:	
$V_{e,2}^{\mu}$	$T \tilde{p}^{\mu  u} \partial_{ u}  u$	$V_{e,2}^i$	$T \tilde{p}^{ij} \partial_j \nu$	
$V_{e,3}^{\mu}$	$T  ilde{P}^{\mu  u} \partial_{ u}  u_n$	$V_{e,3}^i$	$T ilde{p}^{ij}\partial_j u_n$	
:	:	:	:	
$ ilde{S}_{e,1}$	$T \varepsilon^{\mu u ho\sigma} n_{\mu} \zeta_{ u} \partial_{ ho} B_{\sigma}$	$ ilde{S}_{e,1}$	$T\varepsilon^{ijk}\zeta_i\partial_ju_k$	
$ ilde{S}_{e,2}$	$\frac{T}{2}\varepsilon^{\mu\nu\rho\sigma}n_{\mu}\zeta_{\nu}F_{\rho\sigma}$	$\tilde{S}_{e,2}$	$rac{T}{2}arepsilon^{ijk}\zeta_ieta_{jk}$	
:	; #35 po	:	. :	
$\tilde{V}_{e,1}^{\mu}$	$-T\tilde{p}^{\mu}_{\tau}\varepsilon^{\tau\nu\rho\sigma}n_{\nu}\partial_{\rho}B_{\sigma}$	$\tilde{V}_{e,1}^i$	$T\tilde{p}^{i}{}_{l}\varepsilon^{ljk}\partial_{j}u_{k}$	
$\tilde{V}_{e,2}^{e,1}$	$-rac{T}{2} ilde{p}^{\mu}_{\  au}arepsilon^{ au u u}O_{ ho}D_{\sigma}$	$ ilde{V}_{e,2}^{e,1}$	$\frac{T}{2}\tilde{p}^{i}{}_{l}arepsilon^{ljk}eta_{jk}$	
$\tilde{V}_{e,3}^{e,2}$	$T\tilde{p}^{\mu}_{\ \tau}\varepsilon^{\tau\nu\rho\sigma}\zeta_{\nu}\partial_{\rho}B_{\sigma} + (\mu_{s} + \frac{1}{2}\zeta^{\mu}\zeta_{\mu})\tilde{V}^{\mu}_{e,1}$	$\tilde{V}_{e,2}^{e,2}$ $\tilde{V}_{e,3}^{i}$	$-T\left(u^{i}\varepsilon^{jkl}\zeta_{j}\partial_{k}u_{l}-\varepsilon^{ijk}\zeta_{j}\partial_{t}u_{k}\right)$	
' e,3	$\rho = \rho = \sigma + (\mu s + 28 + 8\mu)^{\nu} e, 1$	' e,3	$+(\mu_s + \frac{1}{2}\zeta^k \zeta_k) \tilde{V}_{e,1}^i$	
$ ilde{V}_{e,4}^{\mu}$	$\frac{T}{2}\tilde{p}^{\mu}_{\ \tau}\varepsilon^{\tau\nu\rho\sigma}\zeta_{\nu}F_{\rho\sigma} + (\mu_s + \frac{1}{2}\zeta^{\mu}\zeta_{\mu})\tilde{V}^{\mu}_{e,2}$	$\tilde{V}_{e,4}^i$	$-T\left(u^{i}\frac{1}{2}\varepsilon^{jkl}\zeta_{j}\beta_{kl} + \varepsilon^{ijk}\zeta_{j}e_{k}\right)$	
· e,4	$2r = 5\nu - \rho o + (\mu s + 25 5\mu)^* e, 2$	· e,4	$+(\mu_s + \frac{1}{2}\zeta^k\zeta_k)\tilde{V}_{e,2}^i$	
:	:	:	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	
•	·	<u> </u>	·	

**Table 4:** Independent null superfluid data at the first order in derivatives. Note that we have not, neither do we need to, enlist all the independent data that survives in equilibrium; the ones listed here are the only ones we use in the null superfluid constitutive relations.

where  $\bar{u}^{\mu} = h^{\mu}_{\ \nu} u^{\nu} = u^{\mu} - v^{\mu}$  and  $\bar{\xi}^{\mu} = h^{\mu}_{\ \nu} \xi^{\nu} = \xi^{\mu} - v^{\mu}$ . Various quantities appearing in the constitutive relations can be found via reduction as: fluid densities,

$$\rho = R_n + \sum_{i=1}^{3} \alpha_{R_n,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_n,i} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{\rho} (T f_3 \zeta^{\rho}),$$

$$\epsilon = E + \sum_{i=1}^{3} \alpha_{E,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} - \frac{1}{T} \nabla_{\rho} (T f_1 \zeta^{\rho})$$

$$q = Q + \sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{q,i} - \frac{1}{T} \nabla_{\rho} (T f_2 \zeta^{\rho}),$$

$$s = S + \sum_{i=1}^{3} \alpha_{S,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^2} \nabla_{\rho} (T f_1 \zeta^{\rho}) + \frac{\mu_n}{T^2} \nabla_{\rho} (T f_3 \zeta^{\rho}) + \frac{\mu}{T^2} \nabla_{\rho} (T f_2 \zeta^{\rho}). \quad (4.23)$$

and dissipative currents.

$$\begin{split} \varsigma_{\rho}^{\mu} &= \zeta^{\mu} \left[ \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} \right] - \sum_{i=1}^{3} f_{i} V_{e,i}^{\mu} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{\mu} + \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} \left( Tg_{1} n_{\rho} \zeta_{\sigma} \right), \\ \varsigma_{\epsilon}^{\mu} &= \zeta^{\mu} \left[ \sum_{i=1}^{3} f_{i} S_{5+i} + \left( \mu_{s} + \frac{1}{2} \zeta^{\mu} \zeta_{\mu} \right) \left( \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} \right) - \sum_{i=1}^{5} \beta_{2i} S_{i} \right] \\ &+ \left( \mu_{s} + \frac{1}{2} \zeta^{\mu} \zeta_{\mu} \right) \sum_{i=1}^{3} f_{i} V_{e,i}^{\mu} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i+2}^{\mu} - \sum_{i=1}^{3} \kappa_{1i} V_{i}^{\mu} - \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}_{i}^{\mu} + 3C^{(4)} \mu^{2} B^{\mu} \\ &+ \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} \left( Tg_{3} n_{\rho} \zeta_{\sigma} \right) + C_{1} T^{2} \omega^{\mu}, \\ \varsigma_{t}^{\mu\nu} &= \zeta^{\mu} \zeta^{\nu} \left[ \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \sum_{i=1}^{2} \frac{g_{i}}{2 \hat{\mu}_{s}} \tilde{S}_{e,i} - \sum_{i=1}^{5} \beta_{3i} S_{i} \right] - \eta \sigma^{\mu\nu} - \tilde{\eta} \tilde{\sigma}^{\mu\nu} \\ &- 2 \zeta^{(\mu} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{\nu} + \sum_{i=1}^{3} \kappa_{2i} V_{i}^{\nu} + \sum_{i=1}^{3} \tilde{\kappa}_{2i} \tilde{V}_{i}^{\nu} \right] + \tilde{p}^{\mu\nu} \left[ \sum_{i=1}^{3} f_{i} S_{e,i} - \sum_{i=1}^{5} \beta_{1i} S_{i} \right], \\ \varsigma_{q}^{\mu} &= - \zeta^{\mu} \left[ \sum_{i=1}^{3} \alpha_{R_{s,i}} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} + \sum_{i=1}^{5} \beta_{4i} S_{i} \right] + \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} \left( Tg_{2} n_{\rho} \zeta_{\sigma} \right) \\ &+ \sum_{i=1}^{3} f_{i} V_{e,i}^{\mu} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} + \sum_{i=1}^{5} \beta_{4i} S_{i} \right] + \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} \left( Tg_{2} n_{\rho} \zeta_{\sigma} \right) \\ &+ \sum_{i=1}^{3} f_{i} V_{e,i}^{\mu} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} + \sum_{i=1}^{5} \beta_{4i} S_{i} \right] + \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} \left( Tg_{2} n_{\rho} \zeta_{\sigma} \right) - \frac{1}{T} \partial_{\nu} \left( Tg_{3} n_{\rho} \zeta_{\sigma} \right) \right] \\ &+ \sum_{i=1}^{3} \frac{\mu \kappa_{3i} - \kappa_{1i}}{T} V_{i}^{\mu} + \sum_{i=1}^{3} \frac{\mu \tilde{\kappa}_{3i} - \tilde{\kappa}_{1i}}{T} \tilde{V}_{i}^{\mu} - Tg_{1} \varepsilon^{\mu\nu\rho\sigma} n_{\nu} \zeta_{\rho} \partial_{\sigma} \nu_{n} - Tg_{2} \varepsilon^{\mu\nu\rho\sigma} n_{\nu} \zeta_{\rho} \partial_{\sigma} \nu \\ &+ 2C_{1} T \omega^{\mu}. \end{split}$$

In addition, we also have the Josephson equation,

$$-\frac{1}{2}\zeta^{\mu}\zeta_{\mu} - \mu_{s} + \mu_{n} - \mu = \frac{1}{\beta_{55}}\nabla_{\mu}(R_{s}\xi^{\mu}) - \sum_{i=1}^{4} \frac{\beta_{5i}}{\beta_{55}}S_{i} + \frac{1}{\beta_{55}}\nabla_{\mu}\left(\zeta^{\mu}\sum_{i=1}^{3}\alpha_{R_{s},i}S_{e,i} + \zeta^{\mu}\sum_{i=1}^{2}\tilde{\alpha}_{R_{s},i}\tilde{S}_{e,i} - \sum_{i=1}^{3}f_{i}V_{e,i}^{\mu} - \sum_{i=1}^{2}g_{i}\tilde{V}_{e,i}^{\mu}\right), \quad (4.25)$$

which is the derivative correction of the ideal order version  $\mu_s = -\frac{1}{2}\zeta^{\mu}\zeta_{\mu} - \mu + \mu_n$ . This completes our discussion of the first order Galilean (Newton-Cartan) superfluids; counting of various transport coefficients appearing in the constitutive relations is same as the null superfluid given in §3.5.

#### 4.2 Non-Covariant Notation (for Flat Spacetime)

If the superfluid is coupled to a flat Galilean spacetime, it is fitting to re-express the results in the conventional non-covariant notation where we treat the time and space indices distinctly. It might help the reader to better relate the Galilean superfluid constitutive relations to the existing Galilean literature, e.g. in [20].

Background and Hydrodynamic Fields: On the Newton-Cartan background, we choose a basis  $\{x^{\mu}\}=\{t,x^i\}$  such that the Galilean frame velocity  $(v^{\mu})=\partial_t$ . A flat Galilean background is defined by a particular choice of the Newton-Cartan structure in this basis,

$$n_{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \quad p_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix}, \quad B_{\mu}^{(v)} = 0,$$
 (4.26)

where  $\delta^{ij} = \delta_{ij}$  is the Kronecker delta. It can be checked that the respective Newton-Cartan connection  $\Gamma^{\lambda}_{\mu\nu} = 0$ , justifying the spacetime to be flat. The Newton-Cartan structure in the fluid frame can be worked out from here to be,

$$u^{\mu} = \begin{pmatrix} 1 \\ u^{i} \end{pmatrix}, \quad B_{\mu} = \begin{pmatrix} -\frac{1}{2}u^{k}u_{k} \\ u_{i} \end{pmatrix}, \quad p^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{ij} \end{pmatrix}, \quad p_{\mu\nu} = \begin{pmatrix} u^{k}u_{k} & -u_{j} \\ -u_{i} & \delta_{ij} \end{pmatrix}. \tag{4.27}$$

We define the spatial volume element,

$$\varepsilon^{ijk} = n_{\mu} \varepsilon^{\mu ijk} = \varepsilon^{tijk}. \tag{4.28}$$

The U(1) gauge field  $A_{\mu}$  can be decomposed as  $A_{\mu}dx^{\mu} = A_{t}dt + A_{i}dx^{i}$ . The fluid vorticity and electromagnetic field strength on the other hand can be decomposed as,

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & (\partial_t + u^k \partial_k) u_i + \omega_{ik} u^k \\ -(\partial_t + u^k \partial_k) u_i - \omega_{ik} u^k & \omega_{ij} = \partial_i u_j - \partial_j u_i \end{pmatrix}, \tag{4.29}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -e_i = \partial_t A_i - \partial_i A_t \\ e_i = -\partial_t A_i + \partial_i A_t & \beta_{ij} = \partial_i A_j - \partial_j A_i \end{pmatrix}, \tag{4.30}$$

where  $\omega_{ij}$  is the (dual) spatial vorticity,  $e_i$  is the electric field and  $\beta_{ij}$  is the dual magnetic field. For later use, we define the magnetic field and fluid vorticity,

$$B^{i} = \frac{1}{2}\varepsilon^{ijk}\beta_{jk}, \qquad \omega^{i} = \frac{1}{2}\varepsilon^{ijk}\omega_{jk}. \tag{4.31}$$

Finally the superfluid velocity can be decomposed as,

$$\zeta^{\mu} = \begin{pmatrix} 0 \\ \zeta^{i} \end{pmatrix}, \qquad \xi^{\mu} = \begin{pmatrix} 1 \\ \xi^{i} = u^{i} + \zeta^{i} \end{pmatrix}, \qquad \mu_{s} = -\xi_{t} - \frac{1}{2}\xi^{i}\xi_{i}, \qquad \hat{\mu}_{s} = -\frac{1}{2}\zeta^{i}\zeta_{i}, \qquad (4.32)$$

with the projection operators.

$$\tilde{p}_{\mu\nu} = \begin{pmatrix} u^k u_k & -u_j \\ -u_i & \tilde{p}_{ij} = \delta_{ij} - \frac{\zeta_i \zeta_j}{\zeta^k \zeta_k} \end{pmatrix}, \quad \tilde{p}^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}^{ij} = \delta^{ij} - \frac{\zeta^i \zeta^j}{\zeta^k \zeta_k} \end{pmatrix}. \tag{4.33}$$

**Densities, Currents and Conservation Laws:** In flat spacetime, the conservation laws and the second law of thermodynamics take the well known form,

Mass Conservation: 
$$\partial_t \rho^t + \partial_i \rho^i = 0$$
  
Energy Conservation:  $\partial_t \epsilon^t_{(v)} + \partial_i \epsilon^i_{(v)} = j^i e_i - T_H^{\perp}_t$   
Momentum Conservation:  $\partial_t \rho^j + \partial_i t^{ij}_{(v)} = \left(e^j j^t + \beta^{jk} j_k\right) + T_H^{\perp j},$   
Charge Conservation:  $\partial_t j^t + \partial_i j^i = J_H^{\perp},$   
Second Law of Thermodynamics:  $\partial_t s^t + \partial_i s^i \geq 0,$  (4.34)

where we have identified various Galilean quantities: mass density  $\rho^t$ , mass current  $\rho^i$ , energy density  $\epsilon^t_{(v)}$ , energy current  $\epsilon^i_{(v)}$ , stress tensor  $t^{ij}_{(v)}$ , charge density  $j^t$ , charge current  $j^i$ , entropy density  $s^t$  and entropy current  $s^i$ .

Superfluid Constitutive Relations: Finally, we can read out the structural form of the Galilean superfluid constitutive relations in non-covariant notation using reduction,

$$\rho^{t} = \rho + R_{s}, \qquad \rho^{i} = \rho u^{i} + R_{s} \xi^{i} + \varsigma_{\rho}^{i}, 
\epsilon_{(v)}^{t} = \epsilon + R_{s} \mu_{s} + \frac{1}{2} \rho \vec{u}^{2} + \frac{1}{2} R_{s} \vec{\xi}^{2} + \varsigma_{\rho}^{i} u_{i}, 
\epsilon_{(v)}^{i} = u^{i} \left( \epsilon + P + \frac{1}{2} \rho \vec{u}^{2} + \varsigma_{\rho}^{j} u_{j} \right) + R_{s} \xi^{i} \left( \frac{1}{2} \vec{\xi}^{2} + \mu_{s} \right) + \left( \varsigma_{\epsilon}^{i} + \frac{1}{2} \varsigma_{\rho}^{i} \vec{u}^{2} + \varsigma_{s}^{ij} u_{j} \right), 
t_{(v)}^{ij} = \rho u^{i} u^{j} + R_{s} \xi^{i} \xi^{j} + P \delta^{ij} + \left( \varsigma_{s}^{ij} + 2 \varsigma_{\rho}^{(i} u^{j)} \right), 
t_{(v)}^{t} = q - R_{s}, \qquad t_{s}^{i} = q u^{i} - R_{s} \xi^{i} + \varsigma_{q}^{i}, 
t_{s}^{t} = s, \qquad s^{i} = s u^{i} + \varsigma_{s}^{i}.$$
(4.35)

Various quantities appearing here can also be worked out using reduction: fluid densities,

$$\rho = R_n + \sum_{i=1}^{3} \alpha_{R_n,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_n,i} \tilde{S}_{e,i} - \frac{1}{T} \partial_i (T f_3 \zeta^i),$$

$$\epsilon = E + \sum_{i=1}^{3} \alpha_{E,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} - \frac{1}{T} \partial_i (T f_1 \zeta^i)$$

$$q = Q + \sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{q,i} - \frac{1}{T} \partial_i (T f_2 \zeta^i),$$

$$s = S + \sum_{i=1}^{3} \alpha_{S,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^2} \partial_i (T f_1 \zeta^i) + \frac{\mu_n}{T^2} \partial_i (T f_3 \zeta^i) + \frac{\mu}{T^2} \partial_i (T f_2 \zeta^i),$$
(4.36)

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and dissipative currents.

$$\begin{split} \zeta_{\rho}^{i} &= \zeta^{i} \left[ \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} \right] - \sum_{i=1}^{3} f_{i} V_{e,i}^{i} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{i} + \varepsilon^{ijk} \partial_{j} \left( Tg_{1} \zeta_{k} \right), \\ \zeta_{e}^{i} &= \zeta^{i} \left[ \sum_{i=1}^{3} f_{i} S_{5+i} + \left( \mu_{s} + \frac{1}{2} \zeta^{k} \zeta_{k} \right) \left( \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} \right) - \sum_{i=1}^{5} \beta_{2i} S_{i} \right] \\ &+ \left( \mu_{s} + \frac{1}{2} \zeta^{k} \zeta_{k} \right) \sum_{i=1}^{3} f_{i} V_{e,i}^{i} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i+2}^{i} - \sum_{i=1}^{3} \kappa_{1i} V_{i}^{i} - \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}_{i}^{i} + 3C^{(4)} \mu^{2} B^{i}, \\ &+ \varepsilon^{ijk} \partial_{j} (Tg_{3} \zeta_{k}) + C_{1} T^{2} \omega^{i} \\ \zeta_{i}^{ij} &= \zeta^{i} \zeta^{j} \left[ \sum_{i=1}^{3} \alpha_{R_{s,i}} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} - \sum_{i=1}^{2} \frac{g_{i}}{2 \tilde{\mu}_{s}} \tilde{S}_{e,i} - \sum_{i=1}^{5} \beta_{3i} S_{i} \right] - \eta \sigma^{ij} - \tilde{\eta} \tilde{\sigma}^{ij} \\ &- 2 \zeta^{(i} \left[ \sum_{i=1}^{3} f_{i} V_{e,i}^{j} + \sum_{i=1}^{3} \kappa_{2i} V_{i}^{j} \right] + \sum_{i=1}^{3} \tilde{\kappa}_{2i} \tilde{V}_{i}^{j} \right] + \tilde{p}^{ij} \left[ \sum_{i=1}^{3} f_{i} S_{e,i} - \sum_{i=1}^{5} \beta_{1i} S_{i} \right], \\ \zeta_{q}^{i} &= -\zeta^{i} \left[ \sum_{i=1}^{3} \alpha_{R_{s,i}} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_{s,i}} \tilde{S}_{e,i} + \sum_{i=1}^{5} \beta_{4i} S_{i} \right] + \varepsilon^{ijk} \partial_{j} \left( Tg_{2} \zeta_{k} \right) \\ &+ \sum_{i=1}^{3} f_{i} V_{e,i}^{i} + \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{i} - \sum_{i=1}^{3} \kappa_{3i} V_{i}^{i} - \sum_{i=1}^{3} \tilde{\kappa}_{3i} \tilde{V}_{i}^{i} + 6C^{(4)} \mu B^{i}, \\ \zeta_{s}^{i} &= \zeta^{i} \sum_{i=1}^{5} \frac{\mu \beta_{4i} - \beta_{2i}}{T} S_{i} - \varepsilon^{ijk} \left[ \frac{\mu_{n}}{T} \partial_{j} \left( Tg_{1} \zeta_{k} \right) + \frac{\mu}{T} \partial_{j} \left( Tg_{2} \zeta_{k} \right) - \frac{1}{T} \partial_{j} \left( Tg_{3} \zeta_{k} \right) \right] \\ &+ \sum_{i=1}^{3} \frac{\mu \kappa_{3i} - \kappa_{1i}}{T} V_{i}^{i} + \sum_{i=1}^{3} \frac{\mu \tilde{\kappa}_{3i} - \tilde{\kappa}_{1i}}{T} \tilde{V}_{i}^{i} + Tg_{1} \varepsilon^{ijk} \zeta_{j} \partial_{k} \nu_{n} + Tg_{2} \varepsilon^{ijk} \zeta_{j} \partial_{k} \nu + 2C_{1} T \omega^{i}. \end{split}$$

In addition, we have the Josephson equation,

$$-\frac{1}{2}\zeta^{i}\zeta_{i} - \mu_{s} + \mu_{n} - \mu = \frac{1}{\beta_{55}} \left( \partial_{t}R_{s} + \partial_{i}(R_{s}\xi^{i}) \right) - \sum_{i=1}^{4} \frac{\beta_{5i}}{\beta_{55}} S_{i}$$

$$+ \frac{1}{\beta_{55}} \partial_{k} \left( \zeta^{k} \sum_{i=1}^{3} \alpha_{R_{s},i} S_{e,i} + \zeta^{k} \sum_{i=1}^{2} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,i} - \sum_{i=1}^{3} f_{i} V_{e,i}^{k} - \sum_{i=1}^{2} g_{i} \tilde{V}_{e,i}^{k} \right), \quad (4.38)$$

which is the derivative correction of the ideal order version  $\mu_s = -\frac{1}{2}\zeta^i\zeta_i + \mu_n - \mu$ . These equation can be compared with [20] for which the U(1) chemical potential  $\mu = 0$ . This completes our discussion of Galilean superfluids coupled to flat Galilean spacetime, expressed in non-covariant notation.

### 5 | Discussion

We worked out the most generic constitutive relations of an (anomalous) Galilean superfluid up to first order in derivative expansion, both in parity even and odd sectors. We extended the idea of null fluid introduced in [15, 16] to null superfluid, which is a relativistic embedding of Galilean

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superfluids in one higher dimension, and used these to obtain the mentioned results. We found the spectrum of transport coefficients to be extremely rich with 38 coefficients in parity-even and 13 coefficients in parity-odd sector at first order, in addition to two undetermined constants in parity-odd sector including the U(1) anomaly constant (see table (1)). Out of these, 3 parity-odd and 3 parity-even coefficients survive in equilibrium and determine the hydrostatic physics, while 13 parity-even and 7 parity-odd coefficients govern non-dissipative phenomenon away from equilibrium. On the other hand, 22 parity-even and 3 parity-odd coefficients are dissipative. Though we did not discuss it in the main text, there are hints that 13 parity-even non-dissipative non-hydrostatic coefficients and 3 parity-odd dissipative coefficients vanish on imposing Onsager relations (microscopic reversibility). To avoid confusion with counting, we would like to note that we have removed one parity-even hydrostatic coefficient by redefinition of the U(1) phase  $\varphi$ .

An important point to note is that in this work we have only been interested in a broader class of Galilean systems, and not the non-relativistic ones specifically. A system is said to be Galilean if it respects Galilean symmetry transformations (as opposed to the Poincaré transformations for the relativistic case). On the other hand a non-relativistic system is obtained by taking  $c \to \infty$  limit. Every non-relativistic system is Galilean as it respects Galilean symmetry transformations, but the converse might not be true, i.e. not every Galilean system necessarily follows from  $c \to \infty$  limit of a relativistic system. Keeping this in mind, most of the existing literature on non-relativistic physics (e.g. non-relativistic fluid dynamics in [20]) actually refers to Galilean physics, as it is more natural to formulate a theory with Galilean symmetry, than it is to take a relativistic system and perform a non-relativistic limit (in addition, most of this literature was written before special relativity was well explored). Following this philosophy, here we have focused on Galilean (super)fluids, with the ambition to return to a rigorous analysis of non-relativistic (super)fluids in near future. At this point, we can only conclude that a non-relativistic superfluid obtained as a low energy limit of some relativistic superfluid is at least a subsystem of the Galilean superfluid studied in this paper.

Perhaps the most striking benefit of working in the offshell formalism is that it leads to a complete classification of (super)fluid transport up to all orders in derivative expansion [22–24], and provides a natural setting to attempt writing down a Wilsonian effective action describing the entire (super)fluid dynamics [23, 33–38]. It will be interesting to undertake these ambitious problems in context of null/Galilean (super)fluids, and we plan to return to these in future.

In this paper, we focused on breaking the internal U(1) symmetry of Galilean fluids and obtain a null/Galilean superfluid. The same procedure can also be used to break spacetime symmetries, which lead to the formation of boundaries/surfaces in (super)fluids [39]. In an upcoming paper [40], authors discuss the surface transport for relativistic and Galilean superfluids. Finally, first order computations of this paper can also be easily extended to higher orders; in an ongoing project [41] we are looking at some interesting second order phenomenon in Galilean (super)fluids.

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## A | Equilibrium Partition Function

It was realized by [26, 27] that a huge part of the (super)fluid constitutive relations can be fixed by requiring existence of an equilibrium partition function, which generates the part of the constitutive relations that survive in equilibrium. In this appendix, we will discuss the equilibrium partition function for Galilean superfluids. In hydrodynamics, equilibrium is defined by a set of fields  $\mathcal{K} = \{K^M, \Lambda_K\}$  with  $K^M K_M < 0$ , which act on the background fields  $g_{MN}$ ,  $A_M$  and the superfluid phase  $\varphi$  as an isometry,

$$\delta_{\mathcal{K}}g_{MN} = \nabla_{M}K_{N} + \nabla_{N}K_{M} = 0, \qquad \delta_{\mathcal{K}}A_{M} = \partial_{M}(\Lambda_{K} + K^{N}A_{N}) + K^{N}F_{NM} = 0,$$

$$\delta_{\mathcal{K}}\varphi = K^{M}\partial_{M}\varphi - \Lambda_{K} = K^{M}\xi_{M} - (\Lambda_{K} + K^{N}A_{N}) = 0. \tag{A.1}$$

For simplicity, we choose a basis  $\{x^M\} = \{x^-, t, x^i\}$  such that the null isometry  $\mathcal{V} = \{V = \partial_-, \Lambda_V = 0\}$  and the equilibrium isometry  $\mathcal{K} = \{K = \partial_t, \Lambda_K = 0\}$ . The fact that  $\mathcal{V}$ ,  $\mathcal{K}$  are isometries implies that all the fields are independent of  $x^-$ , t coordinates. In this basis, we decompose the background fields as,

$$ds^{2} = -2e^{-\Phi}(dt + a_{i}dx^{i})(dx^{-} - B_{t}dt - B_{i}dx^{i}) + g_{ij}dx^{i}dx^{j},$$

$$A = -dx^{-} + A_{t}dt + A_{i}dx^{i}.$$
(A.2)

We will denote the covariant derivative associated with the spatial metric  $g_{ij}$  by  $\nabla_i$ . After choosing the said basis, the residual diffeomorphisms are the spatial diffeomorphisms  $x^i \to x^i + \chi^i(x^j)$ , mass gauge transformations  $x^- \to x^- + \chi^-(x^i)$  and Kaluza-Klein gauge transformations  $t \to t + \chi^t(x^i)$ . Under mass gauge transformations, only fields that transform are,

$$\delta_{\chi^{-}}B_{i} = -\partial_{i}\chi^{-}, \qquad \delta_{\chi^{-}}A_{i} = -\partial_{i}\chi^{-},$$
(A.3)

while under Kaluza-Klein gauge transformations,

$$\delta_{\chi^+} a_i = \partial_i \chi^+, \qquad \delta_{\chi^+} B_i = B_t \partial_i \chi^+, \qquad \delta_{\chi^+} A_i = A_t \partial_i \chi^+.$$
 (A.4)

We define the fields,

$$\dot{B}_i = B_i - a_i B_t, \qquad \dot{A}_i = A_i - a_i A_t - \dot{B}_i.$$
(A.5)

 $\acute{B}_{i}$  is mass gauge field which is invariant under Kaluza-Klein gauge transformations.  $\acute{A}_{i}$  on the other hand is invariant under both mass and Kaluza-Klein gauge transformations, and

only transforms under the U(1).  $a_i$  is Kaluza-Klein gauge field. Components of the superfluid velocity  $\xi_M = \partial_M \varphi + A_M$  can be found as,

$$\xi_{-} = -1, \qquad \xi_{t} = A_{t}, \qquad \xi_{i} = \partial_{i} \varphi + A_{i}.$$
 (A.6)

Out of these,  $\xi_i$  is not mass or Kaluza-Klein gauge invariant due to presence of  $A_i$ . We can write an invariant version as,

$$\dot{\xi_i} = \partial_i \varphi + \dot{A_i}. \tag{A.7}$$

The superfluid potential can also be written in terms of these as,

$$\mu_s = -\frac{1}{2} \xi^M \xi_M = -\frac{1}{2} \acute{\xi}^i \acute{\xi}_i - e^{\Phi} A_t + e^{\Phi} B_t, \tag{A.8}$$

and we define  $\dot{\mu}_s = -\frac{1}{2} \dot{\xi}^i \dot{\xi}_i$ . Finally, the fundamental variables at equilibrium are,

$$\Phi, \quad A_t, \quad B_t, \quad a_i, \quad \acute{A}_i, \quad \acute{B}_i, \quad g_{ij}, \quad \varphi.$$
(A.9)

The argument is that at equilibrium, constitutive relations should be derivable from an equilibrium partition function written in terms of these fundamental fields. In covariant terms, variation of an equilibrium partition function W can be parametrized as,

$$\delta W = \int \{ \mathrm{d}x^M \} \sqrt{-g} \left( \frac{1}{2} T^{MN} \delta g_{MN} + J^M \delta A_M + K \delta \varphi \right). \tag{A.10}$$

In our chosen basis it decomposes as,

$$\delta W = \int \left\{ dx^{i} \right\} \sqrt{g_{3}} \left[ \left( T_{t-} + T_{--} B_{t} \right) \delta \Phi + e^{-\Phi} \left( T^{i}_{t} + J^{i} A_{t} \right) \delta a_{i} + \frac{1}{2} e^{-\Phi} T^{ij} \delta g_{ij} + \left( T_{--} \delta B_{t} - e^{-\Phi} \left( T^{i}_{-} - J^{i} \right) \delta \dot{B}_{i} \right) - \left( J_{-} \delta A_{t} - e^{-\Phi} J^{i} \delta \dot{A}_{i} \right) + e^{-\Phi} K \delta \varphi \right], \quad (A.11)$$

where  $g_3 = \det g_{ij}$ . Now, given the most generic partition function  $W[\Phi, A_t, B_t, a_i, \acute{A}_i, \acute{B}_i, g_{ij}, \varphi]$  as a gauge invariant scalar functional of the fundamental fields, various components of the currents  $T^{MN}$ ,  $J^M$ , K can be read out in terms of W as,

$$T_{--} = \frac{1}{\sqrt{g_3}} \frac{\delta W}{\delta B_t}, \qquad T_{t-} = \frac{1}{\sqrt{g_3}} \left( \frac{\delta W}{\delta \Phi} - B_t \frac{\delta W}{\delta B_t} \right),$$

$$T^i_{--} = -\frac{e^{\Phi}}{\sqrt{g_3}} \left( \frac{\delta W}{\delta \dot{B}_i} - \frac{\delta W}{\delta \dot{A}_i} \right), \qquad T^i_{t} = \frac{e^{\Phi}}{\sqrt{g_3}} \left( \frac{\delta W}{\delta a_i} - A_t \frac{\delta W}{\delta \dot{A}_i} \right), \qquad T^{ij} = \frac{2e^{\Phi}}{\sqrt{g_3}} \frac{\delta W}{\delta g_{ij}},$$

$$J_{--} = -\frac{1}{\sqrt{g_3}} \frac{\delta W}{\delta A_t}, \qquad J^i = \frac{e^{\Phi}}{\sqrt{g_3}} \frac{\delta W}{\delta \dot{A}_i}. \tag{A.12}$$

Since these expressions are already in a "non-covariant notation", we can easily perform null reduction to read out the Galilean currents. We define a Galilean frame field to perform the reduction,

$$v_{(K)}^{M} = -\frac{K^{M}}{V_{M}K^{M}} + \frac{K^{R}K_{R}V^{M}}{2(V_{N}K^{N})^{2}} = \begin{pmatrix} e^{\Phi}B_{t} \\ e^{\Phi} \\ 0 \end{pmatrix}.$$
 (A.13)

In  $v_{(K)}^{M}$  Galilean frame, the Galilean currents can be read out in terms of W as,

$$\rho = \frac{1}{\sqrt{g_3}} \frac{\delta W}{\delta B_t}, \qquad \rho^i = \frac{e^{\Phi}}{\sqrt{g_3}} \left( \frac{\delta W}{\delta \dot{B}_i} - \frac{\delta W}{\delta \dot{A}_i} \right), \qquad t_{(v_K)}^{ij} = \frac{2e^{\Phi}}{\sqrt{g_3}} \frac{\delta W}{\delta g_{ij}},$$

$$\epsilon_{(v_K)} = \frac{e^{\Phi}}{\sqrt{g_3}} \frac{\delta W}{\delta \Phi}, \qquad \epsilon_{(v_K)}^i = \frac{e^{2\Phi}}{\sqrt{g_3}} \left( -\frac{\delta W}{\delta a_i} + (A_t - B_t) \frac{\delta W}{\delta \dot{A}_i} + B_t \frac{\delta W}{\delta \dot{B}_i} \right),$$

$$j = \frac{1}{\sqrt{g_3}} \frac{\delta W}{\delta A_t}, \qquad j^i = \frac{e^{\Phi}}{\sqrt{g_3}} \frac{\delta W}{\delta \dot{A}_i}.$$
(A.14)

Finally, we can write down the most general equilibrium partition function W up to first order in derivatives as,

$$W = \int \{\mathrm{d}x^{i}\} \sqrt{g_{3}} \left[ \mathrm{e}^{-\Phi}P + \mathrm{e}^{-\Phi} f_{1} \acute{\xi}^{i} \partial_{i} \Phi + f_{2} \acute{\xi}^{i} \partial_{i} A_{t} + f_{3} \acute{\xi}^{i} \partial_{i} B_{t} + f_{4} \acute{\nabla}_{i} \left( \acute{\xi}^{i} \frac{\partial P}{\partial \mathring{\mu}_{s}} \right) + \acute{\nabla}_{i} (f_{5} \acute{\xi}^{i}) \right.$$

$$\left. + (g_{1} + g_{2}) \varepsilon^{ijk} \acute{\xi}_{i} \partial_{j} \acute{B}_{k} + g_{2} \varepsilon^{ijk} \acute{\xi}_{i} \partial_{j} \acute{A}_{k} + (g_{1} B_{t} + g_{2} A_{t} - \mathrm{e}^{-\Phi} g_{3}) \varepsilon^{ijk} \acute{\xi}_{i} \partial_{j} a_{k} - C_{1} \varepsilon^{ijk} a_{i} \partial_{j} \acute{B}_{k} \right],$$

$$\left. (A.15)$$

where the coefficients P,  $f_i$ ,  $g_i$  are arbitrary functions of the scalars  $\Phi$ ,  $A_t$ ,  $B_t$  and  $\mu_s$ .  $C_1$  on the other hand has to be a constant, so that integral of the term coupling to it is gauge invariant. The term coupling to  $f_4$  is multiplied with the first order equation of motion of  $\varphi$  and hence can be neglected. On the other hand, term coupling to  $f_5$  is a total derivative. Acute reader might note that we have not included a term like to  $C_0\varepsilon^{ijk}\dot{B}_i\partial_j\dot{B}_k$ . The reason is that this term does not have a "covariant analogue" and hence is switched off by the second law of thermodynamics [16]. Finally, this equilibrium partition function does not account for anomalies; for a discussion on anomalous partition function for null fluids see [16, 19].

Varying the partition function W in eqn. (A.15) and using eqn. (A.14), we can read out the equilibrium constitutive relations. We will not perform the explicit variation here, but one can check that the constitutive relations gained are the same as the ones derived in the bulk of the paper, after identifying the equilibrium values of the hydrodynamic fields,

$$u^{M}|_{eqb} = v_{(K)}^{M}, \qquad T|_{eqb} = e^{\Phi}, \qquad \mu_{n}|_{eqb} = e^{\Phi}B_{t}, \qquad \mu|_{eqb} = e^{\Phi}A_{t}.$$
 (A.16)

These can also be summarized as  $\mathcal{B}|_{eqb} = \{\beta^M, \Lambda_\beta\}_{eqb} = \{K^M, \Lambda_K\} = \mathcal{K}$ . Having established that, the equilibrium value of the projected superfluid velocity is given as,

$$\zeta_M|_{eqb} = P_{MN}\xi^N|_{eqb} = \begin{pmatrix} 0\\0\\\xi_i \end{pmatrix}, \tag{A.17}$$

and hence  $\hat{\mu}_s|_{eqb} = \hat{\mu}_s$ . This finishes our discussion of equilibrium partition function for null/Galilean superfluids.

45 B Calculational Details

### B | Calculational Details

In this appendix, we will give details of the computation regarding divergence of the free energy current, glossed over in the main text. We will find the following identities useful in the following computation: let S be a scalar and  $\beta^{\mu}$  be a vector, then,

$$\nabla_{\mu}(\beta^{\mu}S) = \frac{1}{\sqrt{-g}} \mathcal{L}_{\beta} \left( \sqrt{-g}S \right) = \frac{1}{2} S g^{\mu\nu} \mathcal{L}_{\beta} g_{\mu\nu} + \mathcal{L}_{\beta} S. \tag{B.1}$$

There is a corresponding null background version of this identity,

$$\nabla_{M}(\beta^{M}S) = \frac{1}{\sqrt{-g}} \pounds_{\beta} \left( \sqrt{-g}S \right) = \frac{1}{2} S g^{MN} \pounds_{\beta} g_{MN} + \pounds_{\beta} S. \tag{B.2}$$

Given a tensor  $X^{\mu\nu}$ , we have,

$$\nabla_{\mu}\nabla_{\nu}X^{[\mu\nu]} = \frac{1}{2} \left( \nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu} \right) X^{\mu\nu}$$

$$= \frac{1}{2} \left( R_{\mu\nu}{}^{\mu}{}_{\rho}X^{\rho\nu} + R_{\mu\nu}{}^{\nu}{}_{\rho}X^{\mu\rho} \right) = \frac{1}{2} \left( R_{\nu\rho}X^{\rho\nu} - R_{\mu\rho}X^{\mu\rho} \right) = 0.$$
 (B.3)

Similarly,

$$\nabla_M \nabla_N X^{[MN]} = 0. \tag{B.4}$$

Relativistic Superfluid Free Energy Current: Let us start with relativistic superfluids. The  $\delta_{\mathcal{B}}$  variation of hydrodynamic and superfluid fields can be computed to be,

$$\delta_{\mathcal{B}}T = \frac{T}{2}u^{\mu}u^{\nu}\delta_{\mathcal{B}}g_{\mu\nu}, \qquad \delta_{\mathcal{B}}\left(\frac{\mu}{T}\right) = \frac{1}{T}u^{\mu}\delta_{\mathcal{B}}A_{\mu}, \qquad \delta_{\mathcal{B}}\mu_{s} = \frac{1}{2}\xi^{\mu}\xi^{\nu}\delta_{\mathcal{B}}g_{\mu\nu} - \xi^{\mu}\delta_{\mathcal{B}}A_{\mu} - \xi^{\mu}\nabla_{\mu}\delta_{\mathcal{B}}\varphi,$$

$$\delta_{\mathcal{B}}\hat{\mu}_{s} = \frac{1}{2}\left(\zeta^{\mu}\zeta^{\nu} - 2(u^{\rho}\xi_{\rho})u^{(\mu}\zeta^{\nu)}\right)\delta_{\mathcal{B}}g_{\mu\nu} - \zeta^{\mu}\delta_{\mathcal{B}}A_{\mu} - \zeta^{\mu}\nabla_{\mu}\delta_{\mathcal{B}}\varphi,$$

$$\delta_{\mathcal{B}}u^{\mu} = \frac{1}{2}u^{\mu}u^{\rho}u^{\nu}\delta_{\mathcal{B}}g_{\rho\nu}, \qquad \delta_{\mathcal{B}}u_{\mu} = \left(2P_{\mu}{}^{(\rho}u^{\nu)} - u_{\mu}u^{\rho}u^{\nu}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\nu},$$

$$\delta_{\mathcal{B}}\zeta^{\mu} = \left(u^{\mu}u^{(\rho}\zeta^{\sigma)} - P^{\mu(\rho}\xi^{\sigma)}\right)\delta_{\mathcal{B}}g_{\rho\sigma} + P^{\mu\nu}\delta_{\mathcal{B}}\xi_{\nu}, \quad \delta_{\mathcal{B}}\zeta_{\mu} = (u^{\nu}\xi_{\nu})u^{(\rho}P_{\mu}^{\sigma)}\delta_{\mathcal{B}}g_{\rho\sigma} + P_{\mu}{}^{\nu}\delta_{\mathcal{B}}\xi_{\nu}. \quad (B.5)$$

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The first order parity-even free energy current  $\mathcal{N}^{\mu}$  in eqn. (2.30) has a term  $2f_1u^{[\mu}\xi^{\nu]}\frac{1}{T^2}\partial_{\nu}T$ . We compute its divergence,

$$\nabla_{\mu} \left( 2f_{1}u^{[\mu}\zeta^{\nu]} \frac{1}{T^{2}} \partial_{\nu}T \right) = f_{1}\zeta^{\nu} \frac{1}{2T} \partial_{\nu}Tg^{\rho\sigma} \delta_{\mathbb{B}}g_{\rho\sigma} + \delta_{\mathbb{B}} \left( f_{1}\zeta^{\nu} \frac{1}{T} \partial_{\nu}T \right) - \nabla_{\mu} \left( f_{1}\zeta^{\mu} \frac{1}{T} \delta_{\mathbb{B}}T \right)$$

$$= f_{1}\zeta^{\nu} \frac{1}{2T} \partial_{\nu}TP^{\rho\sigma} \delta_{\mathbb{B}}g_{\rho\sigma} + f_{1}\frac{1}{T} \partial_{\nu}T\delta_{\mathbb{B}}\zeta^{\nu} + \zeta^{\nu} \frac{1}{T} \partial_{\nu}T\delta_{\mathbb{B}}f_{1}$$

$$- f_{1}\zeta^{\nu} \frac{1}{2T} \partial_{\nu}Tu^{\rho}u^{\sigma} \delta_{\mathbb{B}}g_{\rho\sigma} - f_{1}\zeta^{\nu} \frac{1}{T^{2}} \partial_{\nu}T\delta_{\mathbb{B}}T + f_{1}\zeta^{\nu} \frac{1}{T} \partial_{\nu}\delta_{\mathbb{B}}T - \nabla_{\mu} \left( f_{1}\zeta^{\mu} \frac{1}{T} \delta_{\mathbb{B}}T \right)$$

$$= f_{1}\zeta^{\nu} \frac{1}{2T} \partial_{\nu}TP^{\rho\sigma} \delta_{\mathbb{B}}g_{\rho\sigma} + f_{1}\frac{1}{T} \partial_{\nu}T \left[ \left( u^{\nu}u^{(\rho}\zeta^{\sigma)} - P^{\nu(\rho}\xi^{\sigma)} \right) \delta_{\mathbb{B}}g_{\rho\sigma} + P^{\nu\rho}\delta_{\mathbb{B}}\xi_{\rho} \right]$$

$$+ \zeta^{\nu} \frac{1}{T} \partial_{\nu}T \left( \frac{\partial f_{1}}{\partial T} \delta_{\mathbb{B}}T + \frac{\partial f_{1}}{\partial \nu} \delta_{\mathbb{B}}\nu + \frac{\partial f_{1}}{\partial \hat{\mu}_{s}} \delta_{\mathbb{B}}\hat{\mu}_{s} \right)$$

$$- f_{1}\zeta^{\nu} \frac{1}{2T} \partial_{\nu}Tu^{\rho}u^{\sigma}\delta_{\mathbb{B}}g_{\rho\sigma} - \nabla_{\mu} \left( f_{1}\zeta^{\mu} \right) \frac{1}{T} \delta_{\mathbb{B}}T$$

$$= \left[ u^{\rho}u^{\sigma} \left( \alpha_{E,1}S_{e,1} - \frac{1}{T}\nabla_{\mu} \left( Tf_{1}\zeta^{\mu} \right) \right) + \left( \zeta^{\rho}\zeta^{\sigma} - 2(u^{\mu}\xi_{\mu})u^{(\rho}\zeta^{\sigma)} \right) S_{e,1}\alpha_{R_{s},1}$$

$$+ \tilde{P}^{\rho\sigma}f_{1}S_{e,1} + 2u^{(\rho}\zeta^{\sigma)}f_{1}S_{5} - f_{1}2\xi^{(\rho}V_{e,1}^{\sigma)} \right] \frac{1}{2}\delta_{\mathbb{B}}g_{\rho\sigma}$$

$$+ \left[ u^{\rho}\alpha_{Q,1}S_{e,1} + f_{1}V_{e,1}^{\rho} - \zeta^{\rho}\alpha_{R_{s,1}}S_{e,1} \right] \delta_{\mathbb{B}}A_{\rho} + \left[ f_{1}V_{e,1}^{\rho} - \zeta^{\rho}\alpha_{R_{s,1}}S_{e,1} \right] \partial_{\rho}\delta_{\mathbb{B}}\varphi. \tag{B.6}$$

Performing a differentiation by parts,

$$\nabla_{\mu} \left( 2f_{1}u^{[\mu}\zeta^{\nu]} \frac{1}{T^{2}} \partial_{\nu}T + \mathcal{O}(\partial^{2}) \right) 
= \left[ u^{\rho}u^{\sigma} \left( \alpha_{E,1}S_{e,1} - \frac{1}{T} \nabla_{\mu} \left( Tf_{1}\zeta^{\mu} \right) \right) + \left( \zeta^{\rho}\zeta^{\sigma} - 2(u^{\mu}\xi_{\mu})u^{(\rho}\zeta^{\sigma)} \right) S_{e,1}\alpha_{R_{s},1} \right. 
+ \tilde{P}^{\rho\sigma}f_{1}S_{e,1} + 2u^{(\rho}\zeta^{\sigma)}f_{1}S_{5} - f_{1}2\xi^{(\rho}V_{e,1}^{\sigma)} \right] \frac{1}{2} \delta_{\mathbb{B}}g_{\rho\sigma} 
+ \left[ u^{\rho}\alpha_{Q,1}S_{e,1} + f_{1}V_{e,1}^{\rho} - \zeta^{\rho}\alpha_{R_{s},1}S_{e,1} \right] \delta_{\mathbb{B}}A_{\rho} - \nabla_{\rho} \left[ f_{1}V_{e,1}^{\rho} - \zeta^{\rho}\alpha_{R_{s},1}S_{e,1} \right] \delta_{\mathbb{B}}\varphi. \quad (B.7)$$

From here we can read out the contributions to the constitutive relations eqn. (2.32). Similarly divergence of the other term in eqn. (2.30) coupling to  $f_2$  can also be computed. Now, the first order parity-odd free energy current  $\mathcal{N}^{\mu}$  in eqn. (2.42) has a term  $g_2\beta^{\mu}\tilde{S}_{e,2} + g_2\tilde{V}_2^{\mu}$ . We can

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compute its divergence as,

$$\begin{split} \nabla_{\mu} \left( g_{2} \beta^{\mu} \tilde{S}_{e,2} + g_{2} \tilde{V}_{2}^{\mu} \right) &= \epsilon^{\tau \nu \rho \sigma} \delta_{\mathbb{B}} \left( g_{2} T \frac{1}{2} \xi_{\tau} u_{\nu} F_{\rho \sigma} \right) - \nabla_{\mu} \left( \epsilon^{\mu \tau \nu \sigma} g_{2} T \xi_{\tau} u_{\nu} \delta_{\mathbb{B}} A_{\sigma} \right) \\ &= \frac{T}{2} \epsilon^{\tau \nu \rho \sigma} \xi_{\tau} u_{\nu} F_{\rho \sigma} \delta_{\mathbb{B}} g_{2} + \frac{1}{2} g_{2} \epsilon^{\tau \nu \rho \sigma} \xi_{\tau} u_{\nu} F_{\rho \sigma} \delta_{\mathbb{B}} T + \epsilon^{\tau \nu \rho \sigma} g_{2} T \frac{1}{2} \xi_{\tau} F_{\rho \sigma} \delta_{\mathbb{B}} u_{\nu} \\ &+ \frac{T}{2} g_{2} \epsilon^{\tau \nu \rho \sigma} u_{\nu} F_{\rho \sigma} \delta_{\mathbb{B}} \xi_{\tau} + \epsilon^{\tau \nu \rho \sigma} g_{2} T \xi_{\tau} u_{\nu} \nabla_{\rho} \delta_{\mathbb{B}} A_{\sigma} - \nabla_{\rho} \left( \epsilon^{\tau \nu \rho \sigma} g_{2} T \xi_{\tau} u_{\nu} \delta_{\mathbb{B}} A_{\sigma} \right) \\ &= \frac{T}{2} \epsilon^{\tau \nu \rho \sigma} \xi_{\tau} u_{\nu} F_{\rho \sigma} \left( \frac{\partial g_{2}}{\partial T} \delta_{\mathbb{B}} T + \frac{\partial g_{2}}{\partial \nu} \delta_{\mathbb{B}} \nu + \frac{\partial g_{2}}{\partial \hat{\mu}_{s}} \delta_{\mathbb{B}} \hat{\mu}_{s} \right) + \epsilon^{\tau \nu \rho \sigma} g_{2} T \frac{1}{2} \xi_{\tau} F_{\rho \sigma} 2 P_{\nu}^{\ (\rho} u^{\sigma)} \frac{1}{2} \delta_{\mathbb{B}} g_{\rho \sigma} \\ &+ \left( g_{2} T \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} u_{\nu} F_{\rho \sigma} - \nabla_{\rho} \left( \epsilon^{\rho \mu \tau \nu} g_{2} T \xi_{\tau} u_{\nu} \right) \right) \delta_{\mathbb{B}} A_{\mu} + g_{2} T \frac{1}{2} \epsilon^{\tau \nu \rho \sigma} u_{\nu} F_{\rho \sigma} \nabla_{\tau} \delta_{\mathbb{B}} \varphi \\ &= \left[ u^{\mu} u^{\nu} \tilde{\alpha}_{E,2} \tilde{S}_{e,2} + 2 g_{2} u^{(\mu} \tilde{V}_{e,4}^{\nu)} + \left( \zeta^{\mu} \zeta^{\nu} - 2 (u^{\rho} \xi_{\rho}) u^{(\mu} \zeta^{\nu)} \right) \tilde{\alpha}_{Rs,i} \tilde{S}_{e,2} - \zeta^{\mu} \zeta^{\nu} \frac{g_{2}}{2 \hat{\mu}_{s}} \tilde{S}_{e,2} \right] \frac{1}{2} \delta_{\mathbb{B}} g_{\mu \nu} \\ &+ \left[ u^{\mu} \tilde{\alpha}_{Q,2} \tilde{S}_{e,2} + g_{2} V_{e,2}^{\mu} - \zeta^{\mu} \tilde{\alpha}_{Rs,2} \tilde{S}_{e,2} - \nabla_{\rho} \left( \epsilon^{\rho \mu \tau \nu} g_{2} T \xi_{\tau} u_{\nu} \right) \right] \delta_{\mathbb{B}} A_{\mu} \\ &+ \left[ g_{2} V_{e,2}^{\mu} - \zeta^{\mu} \tilde{\alpha}_{Rs,i} \tilde{S}_{e,2} \right] \nabla_{\mu} \delta_{\mathbb{B}} \varphi. \end{split} \tag{B.8}$$

Performing a differentiation by parts,

$$\nabla_{\mu} \left( g_{2} u^{\mu} \tilde{S}_{e,2} + g_{2} \tilde{V}_{2} + \mathcal{O}(\partial^{2}) \right)$$

$$= \left[ u^{\mu} u^{\nu} \tilde{\alpha}_{E,2} \tilde{S}_{e,2} + 2g_{2} u^{(\mu} \tilde{V}_{e,4}^{\nu)} + \left( \zeta^{\mu} \zeta^{\nu} - 2(u^{\rho} \xi_{\rho}) u^{(\mu} \zeta^{\nu)} \right) \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,2} - \zeta^{\mu} \zeta^{\nu} \frac{g_{2}}{2\hat{\mu}_{s}} \tilde{S}_{e,2} \right] \frac{1}{2} \delta_{\mathbb{B}} g_{\mu\nu}$$

$$+ \left[ u^{\mu} \tilde{\alpha}_{Q,2} \tilde{S}_{e,2} + g_{2} V_{e,2}^{\mu} - \zeta^{\mu} \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} - \nabla_{\rho} \left( \epsilon^{\rho\mu\tau\nu} g_{2} T \xi_{\tau} u_{\nu} \right) \right] \delta_{\mathbb{B}} A_{\mu}$$

$$- \nabla_{\mu} \left[ g_{2} V_{e,2}^{\mu} - \zeta^{\mu} \tilde{\alpha}_{R_{s},i} \tilde{S}_{e,2} \right] \delta_{\mathbb{B}} \varphi. \tag{B.9}$$

From here we can read out the contributions to the constitutive relations eqn. (2.43). Similarly divergence of the other term in eqn. (2.42) coupling to  $g_1$  can also be computed. There is another term in the parity-odd free energy current  $C_1T^2\omega^{\mu}$ ; its divergence is given as,

$$\nabla_{\mu} \left( C_1 T^2 \omega^{\mu} \right) = -2C_1 T \epsilon^{\mu\nu\rho\sigma} u_{\mu} \partial_{\nu} T \partial_{\rho} u_{\sigma} + C_1 T^2 \epsilon^{\mu\nu\rho\sigma} \partial_{\mu} u_{\nu} \partial_{\rho} u_{\sigma}$$
$$= 2C_1 T^3 \omega^{(\mu} u^{\nu)} \delta_{\mathcal{B}} g_{\mu\nu}. \tag{B.10}$$

This can be matched with the constitutive relations eqn. (2.43).

Null Superfluid Free Energy Current: We now move on to superfluids. The  $\delta_{\mathcal{B}}$  variation of hydrodynamic and superfluid fields can be computed to be,

$$\begin{split} \delta_{\mathbb{B}}T &= TV^{(M}u^N)\delta_{\mathbb{B}}g_{MN}, \qquad \delta_{\mathbb{B}}\nu_n = \frac{1}{2T}u^Mu^N\delta_{\mathbb{B}}g_{MN}, \qquad \delta_{\mathbb{B}}\nu = \frac{1}{T}u^M\delta_{\mathbb{B}}A_M, \\ \delta_{\mathbb{B}}\mu_s &= \frac{1}{2}\xi^M\xi^N\delta_{\mathbb{B}}g_{MN} - \xi^M\delta_{\mathbb{B}}A_M - \xi^M\nabla_M\delta_{\mathbb{B}}\varphi, \\ \delta_{\mathbb{B}}\hat{\mu}_s &= \frac{1}{2}\left(\zeta^M\zeta^N + 2\zeta^{(M}u^N) - 2\zeta^{(M}V^N)(u^R\xi_R)\right)\delta_{\mathbb{B}}g_{MN} - \zeta^M\delta_{\mathbb{B}}A_M - \zeta^M\nabla_M\delta_{\mathbb{B}}\varphi, \\ \delta_{\mathbb{B}}u^M &= \left(2u^MV^{(R}u^S) + V^Mu^Ru^S\right)\frac{1}{2}\delta_{\mathbb{B}}g_{RS}, \quad \delta_{\mathbb{B}}u_M = \left(2P_M^{\ \ (R}u^S) - V_Mu^Ru^S\right)\frac{1}{2}\delta_{\mathbb{B}}g_{RS}, \end{split}$$

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$$\delta_{\mathcal{B}}\zeta^{M} = \left(-2\xi^{(R}P^{S)M} + 2\zeta^{(R}V^{S)}u^{M} + 2\zeta^{(R}u^{S)}V^{M}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{RS} + P^{MN}\delta_{\mathcal{B}}\xi_{N},$$

$$\delta_{\mathcal{B}}\zeta_{M} = \left(2(u^{N}\xi_{N})P_{M}^{\ \ (R}V^{S)} - 2P_{M}^{\ \ (R}u^{S)}\right)\frac{1}{2}\delta_{\mathcal{B}}g_{RS} + P_{M}^{\ \ N}\delta_{\mathcal{B}}\xi_{N}.$$
(B.11)

The first order parity-even free energy current  $\mathcal{N}^M$  in eqn. (3.24) has a term  $2f_1u^{[M}\zeta^{N]}\frac{1}{T^2}\partial_N T$ . We compute its divergence,

$$\begin{split} \nabla_{M} \left( 2f_{1}u^{[M}\zeta^{N]} \frac{1}{T^{2}} \partial_{N}T \right) &= f_{1}\zeta^{N} \frac{1}{2T} \partial_{N}Tg^{RS} \delta_{\mathcal{B}}g_{RS} + \delta_{\mathcal{B}} \left( f_{1} \frac{1}{T}\zeta^{N} \partial_{N}T \right) - \nabla_{M} \left( f_{1}\zeta^{M} \frac{1}{T} \delta_{\mathcal{B}}T \right) \\ &= f_{1}\zeta^{N} \frac{1}{2T} \partial_{N}TP^{RS} \delta_{\mathcal{B}}g_{RS} + f_{1} \frac{1}{T} \partial_{N}T \delta_{\mathcal{B}}\zeta^{N} + \frac{1}{T}\zeta^{N} \partial_{N}T \delta_{\mathcal{B}}f_{1} \\ &- f_{1}\zeta^{N} \frac{1}{T} \partial_{N}TV^{R}u^{S} \delta_{\mathcal{B}}g_{RS} - f_{1} \frac{1}{T^{2}}\zeta^{N} \partial_{N}T \delta_{\mathcal{B}}T + f_{1} \frac{1}{T}\zeta^{N} \partial_{N}\delta_{\mathcal{B}}T - \nabla_{M} \left( f_{1}\zeta^{M} \frac{1}{T}\delta_{\mathcal{B}}T \right) \\ &= f_{1}\zeta^{N} \frac{1}{2T} \partial_{N}TP^{RS} \delta_{\mathcal{B}}g_{RS} + f_{1} \frac{1}{T} \partial_{M}T \left[ \left( -2\xi^{(R}P^{S)M} + 2\zeta^{(R}V^{S)}u^{M} \right) \frac{1}{2} \delta_{\mathcal{B}}g_{RS} + P^{MN}\delta_{\mathcal{B}}\xi_{N} \right] \\ &+ \frac{1}{T}\zeta^{N} \partial_{N}T \left( \frac{\partial f_{1}}{\partial T}\delta_{\mathcal{B}}T + \frac{\partial f_{1}}{\partial \nu}\delta_{\mathcal{B}}\nu + \frac{\partial f_{1}}{\partial \nu_{n}}\delta_{\mathcal{B}}\nu_{n} + \frac{\partial f_{1}}{\partial \hat{\mu}_{s}}\delta_{\mathcal{B}}\hat{\mu}_{s} \right) \\ &- f_{1}\zeta^{N} \frac{1}{T}\partial_{N}TV^{R}u^{S}\delta_{\mathcal{B}}g_{RS} - \nabla_{M} \left( f_{1}\zeta^{M} \right) \frac{1}{T}\delta_{\mathcal{B}}T \\ &= \left[ 2V^{(R}u^{S)} \left( \alpha_{E,1}S_{e,1} - \frac{1}{T}\nabla_{M} \left( Tf_{1}\zeta^{M} \right) \right) + u^{R}u^{S}\alpha_{R_{n},1}S_{e,1} + \tilde{P}^{RS}f_{1}S_{e,1} - 2f_{1}\xi^{(R}V_{e,1}^{S)} \right. \\ &+ \left. \left( \zeta^{R}\zeta^{S} + 2\zeta^{(R}u^{S)} - 2\zeta^{(R}V^{S)}(u^{M}\xi_{M}) \right) \alpha_{R_{s},1}S_{e,1} + 2\zeta^{(R}V^{S)}f_{1}S_{6} \right] \frac{1}{2}\delta_{\mathcal{B}}g_{RS} \\ &+ \left[ u^{M}\alpha_{Q,1}S_{e,1} - \zeta^{M}\alpha_{R_{s},1}S_{e,1} + f_{1}V_{e,1}^{M} \right] \delta_{\mathcal{B}}A_{M} + \left[ f_{1}V_{e,1}^{M} - \zeta^{M}\alpha_{R_{s},1}S_{e,1} \right] \nabla_{M}\delta_{\mathcal{B}}\varphi. \end{split}$$
 (B.12)

Performing a differentiation by parts,

$$\nabla_{M} \left( 2f_{1}u^{[M}\zeta^{N]} \frac{1}{T^{2}} \partial_{N}T + \mathcal{O}(\partial^{2}) \right) 
= \left[ 2V^{(R}u^{S)} \left( \alpha_{E,1}S_{e,1} - \frac{1}{T} \nabla_{M} \left( Tf_{1}\zeta^{M} \right) \right) + u^{R}u^{S} \alpha_{R_{n},1}S_{e,1} + \tilde{P}^{RS}f_{1}S_{e,1} - 2f_{1}\xi^{(R}V_{e,1}^{S)} \right. 
\left. + \left( \zeta^{R}\zeta^{S} + 2\zeta^{(R}u^{S)} - 2\zeta^{(R}V^{S)}(u^{M}\xi_{M}) \right) \alpha_{R_{s},1}S_{e,1} + 2\zeta^{(R}V^{S)}f_{1}S_{6} \right] \frac{1}{2} \delta_{\mathbb{B}}g_{RS} 
+ \left[ u^{M}\alpha_{Q,1}S_{e,1} - \zeta^{M}\alpha_{R_{s},1}S_{e,1} + f_{1}V_{e,1}^{M} \right] \delta_{\mathbb{B}}A_{M} + \nabla_{M} \left[ \zeta^{M}\alpha_{R_{s},1}S_{e,1} - f_{1}V_{e,1}^{M} \right] \delta_{\mathbb{B}}\varphi.$$
(B.13)

From here we can read out the contributions to the constitutive relations eqn. (3.26). Similarly divergence of the other terms in eqn. (3.24) coupling to  $f_2$ ,  $f_3$  can also be computed. Now, the first order parity-odd free energy current  $\mathcal{N}^M$  in eqn. (3.36) has a term  $g_2\beta^M\tilde{S}_{e,2} + g_2\tilde{V}_3^M$ . We

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can compute its divergence as,

$$\begin{split} \nabla_{M} \left( g_{2} \beta^{M} \tilde{S}_{e,2} + g_{2} \tilde{V}_{3}^{M} \right) &= \frac{1}{2} \epsilon^{NRSTK} \delta_{\mathbb{B}} \left( g_{2} T \xi_{N} V_{R} u_{S} F_{TK} \right) - \nabla_{T} \left( \epsilon^{NRSTK} g_{2} T \xi_{N} V_{R} u_{S} \delta_{\mathbb{B}} A_{K} \right) \\ &= \frac{1}{2} \epsilon^{NRSTK} T \xi_{N} V_{R} u_{S} F_{TK} \delta_{\mathbb{B}} g_{2} + \frac{1}{2} \epsilon^{NRSTK} g_{2} T \xi_{N} V_{R} F_{TK} \delta_{\mathbb{B}} u_{S} + \frac{1}{2} \epsilon^{NRSTK} g_{2} \xi_{N} V_{R} u_{S} F_{TK} \delta_{\mathbb{B}} T \\ &\quad + \frac{1}{2} \epsilon^{NRSTK} g_{2} T \xi_{N} u_{S} F_{TK} \delta_{\mathbb{B}} V_{R} + \frac{1}{2} \epsilon^{NRSTK} g_{2} T V_{R} u_{S} F_{TK} \delta_{\mathbb{B}} \xi_{N} \\ &\quad + \epsilon^{NRSTK} g_{2} T \xi_{N} V_{R} u_{S} \nabla_{T} \delta_{\mathbb{B}} A_{K} - \nabla_{T} \left( \epsilon^{NRSTK} g_{2} T \xi_{N} V_{R} u_{S} \delta_{\mathbb{B}} A_{K} \right) \\ &= \frac{1}{2} \epsilon^{NRSTK} T \xi_{N} V_{R} u_{S} F_{TK} \left( \frac{\partial g_{2}}{\partial T} \delta_{\mathbb{B}} T + \frac{\partial g_{2}}{\partial \nu} \delta_{\mathbb{B}} \nu + \frac{\partial g_{2}}{\partial \nu} \delta_{\mathbb{B}} \nu_{n} + \frac{\delta g_{2}}{\delta \hat{\mu}_{s}} \delta_{\mathbb{B}} \hat{\mu}_{s} \right) \\ &\quad - u^{A} P_{M}^{B} g_{2} T \frac{1}{2} \epsilon^{MNRTK} V_{N} u_{R} F_{TK} \delta_{\mathbb{B}} g_{AB} + \frac{1}{2} \epsilon^{NRSTK} g_{2} T \xi_{N} u_{S} F_{TK} P_{R}^{B} V^{A} \delta_{\mathbb{B}} g_{AB} \\ &\quad - \nabla_{T} \left( \epsilon^{TMNRS} g_{2} T \xi_{N} V_{R} u_{S} \right) \delta_{\mathbb{B}} A_{M} + \frac{1}{2} \epsilon^{NRSTK} g_{2} T V_{R} u_{S} F_{TK} \delta_{\mathbb{B}} \xi_{N} \\ &= \left[ 2 \tilde{\alpha}_{E,2} V^{(M} u^{N)} \tilde{S}_{e,2} + \tilde{\alpha}_{R_{n},2} u^{M} u^{N} \tilde{S}_{e,2} - 2 g_{2} u^{(M} \tilde{V}_{e,2}^{N)} - 2 g_{2} V^{(M} \tilde{V}_{e,4}^{N)} - \zeta^{M} \zeta^{N} \frac{g_{2}}{2 \hat{\mu}_{s}} \tilde{S}_{e,2} \right. \\ &\quad + \left. \left( \zeta^{M} \zeta^{N} + 2 \zeta^{(M} u^{N)} - 2 \zeta^{(M} V^{N)} (u^{R} \xi_{R}) \right) \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} \right] \frac{1}{2} \delta_{\mathbb{B}} g_{MN} \\ &\quad + \left[ u^{M} \tilde{\alpha}_{Q,2} \tilde{S}_{e,2} + g_{2} \tilde{V}_{e,2} - \zeta^{M} \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} - P_{K}^{M} \nabla_{T} \left( \epsilon^{TKNRS} g_{2} T \xi_{N} V_{R} u_{S} \right) \right] \delta_{\mathbb{B}} A_{M} \\ &\quad + \left[ g_{2} \tilde{V}_{e,2} - \zeta^{M} \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} \right] \nabla_{M} \delta_{\mathbb{B}} \varphi. \end{split} \tag{B.14}$$

Performing a differentiation by parts,

$$\nabla_{M} \left( g_{2} \beta^{M} \tilde{S}_{e,2} + g_{2} \tilde{V}_{3}^{M} + \mathcal{O}(\partial^{2}) \right) 
= \left[ 2V^{(M} u^{N)} \tilde{\alpha}_{E,2} \tilde{S}_{e,2} + u^{M} u^{N} \tilde{\alpha}_{R_{n},2} \tilde{S}_{e,2} - 2g_{2} u^{(M} \tilde{V}_{e,2}^{N)} - 2g_{2} V^{(M} \tilde{V}_{e,4}^{N)} - \zeta^{M} \zeta^{N} \frac{g_{2}}{2\hat{\mu}_{s}} \tilde{S}_{e,2} \right. 
+ \left. \left( \zeta^{M} \zeta^{N} + 2\zeta^{(M} u^{N)} - 2\zeta^{(M} V^{N)} (u^{R} \xi_{R}) \right) \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} \right] \frac{1}{2} \delta_{\mathbb{B}} g_{MN} 
+ \left. \left[ u^{M} \tilde{\alpha}_{Q,2} \tilde{S}_{e,2} + g_{2} \tilde{V}_{e,2} - \zeta^{M} \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} - P_{K}^{M} \nabla_{T} \left( \epsilon^{TKNRS} g_{2} T \xi_{N} V_{R} u_{S} \right) \right] \delta_{\mathbb{B}} A_{M} 
+ \nabla_{M} \left[ \zeta^{M} \tilde{\alpha}_{R_{s},2} \tilde{S}_{e,2} - g_{2} \tilde{V}_{e,2} \right] \delta_{\mathbb{B}} \varphi.$$
(B.15)

From here we can read out the contributions to the constitutive relations eqn. (3.37). Similarly divergence of the other term in eqn. (3.36) coupling to  $g_1$  can also be computed. Divergence of the term coupling to  $g_3$  is particularly simple,

$$\nabla_{M} \left( g_{3} \tilde{V}_{1}^{M} \right) = \nabla_{M} \left( \frac{g_{3}}{T} \epsilon^{MNRST} V_{N} u_{R} \zeta_{S} \partial_{T} T \right)$$

$$= -\nabla_{M} \left( g_{3} T \epsilon^{MNRST} V_{N} u_{R} \zeta_{S} \right) \partial_{T} \left( \frac{1}{T} \right)$$

$$= V^{(M} P_{P}^{N)} \nabla_{K} \left( g_{3} T \epsilon^{PKRST} V_{R} u_{S} \zeta_{T} \right) \delta_{\mathcal{B}} g_{MN}. \tag{B.16}$$

Finally the last term in parity-odd free energy current  $C_1T\omega^M$  has divergence,

$$\nabla_M \left( C_1 T \omega^M \right) = C_1 T^2 \omega^{(M} V^{N)} \delta_{\mathcal{B}} g_{MN}. \tag{B.17}$$

This can be matched with the constitutive relations eqn. (3.37).

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