12

Beam Deflections by Discontinuity Functions

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§12.1. Discontinuity Functions

As the name indicates, Discontinuity Functions (abbreviation: D.F.) were invented to compactly represent discontinuities of various kinds in mathematical functions.[1] They may be found under several names and notations in other fields such as Physics, Chemistry and Electrical Engineering, as well as Fluid Dynamics. In this course they will be used to represent beam x-functions that range from applied loads through deflections. We will adopt MacAuley's angle-brackets notation, which is that used by most Mechanics of Materials textbooks.

The generic nomenclature for these functions, taking x as independent variable, is

$$\langle x - a \rangle^n \tag{12.1}$$

in which a denotes the position along x where a discontinuity occurs. Superscript n is an integer that characterizes the kind of discontinuity represented by (12.1). (This integer can be interpreted as an exponent if nonnegative, as discussed below.) The Table provided in Figure 12.1 lists the most widely used D.F., along with their definitions, and a useful integration rule.

A Short Table of Discontinuity Functions

Name	Symbol	Definition
doublet	$< x{-}a >^{-2}$	antiderivative of doublet function is delta function
[Dirac] delta	$< x{-}a >^{-1}$	antiderivative of delta function is step function
step	$< x{-}a >^{0}$	1 if $x > a$, else 0
ramp	$< x{-}a >^{1}$	$x{-}a$ if $x > a$, else 0
parabolic ramp	$< x{-}a >^{2}$	$(x{-}a)^2$ if $x > a$, else 0
....		
n^{th} order ramp	$< x{-}a >^{n}$	$(x{-}a)^n$ if $x > a$, else 0 ($n \ge 0$)

Useful integration formula for $n = 0, 1, 2, \ldots$

$$\int_{x_0}^{x} < x - a >^n \, dx = \frac{< x - a >^{n+1}}{n+1} \qquad \text{valid if } n \ge 0 \text{ and } x_0 \le a$$

For beam problems where origin of x is at left end, x_0 is normally 0

If $n = -1$, integral is step function. If $n = -2$, integral is delta function.

FIGURE 12.1. A list of Discontinuity Functions. This Table is also provided on the Supplementary Crib Sheet for Midterm Exam #3.

It is convenient to distinguish between two types of D.F.: nonsingular and singular.

[1] Introduced, with the present notation, by the British mathematical MacAuley in the late XIX Century. They are called "MacAuley functions" in some textbooks.

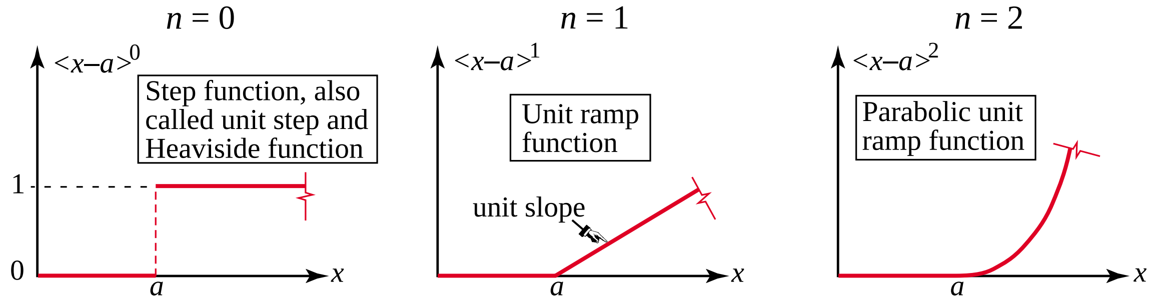


FIGURE 12.2. Discontinuity Functions for nonegative exponents. Pictured are cases $n = 0$, 1 and 2. Note that for $n = 0$ the function receives several names: step, unit step, and Heaviside.

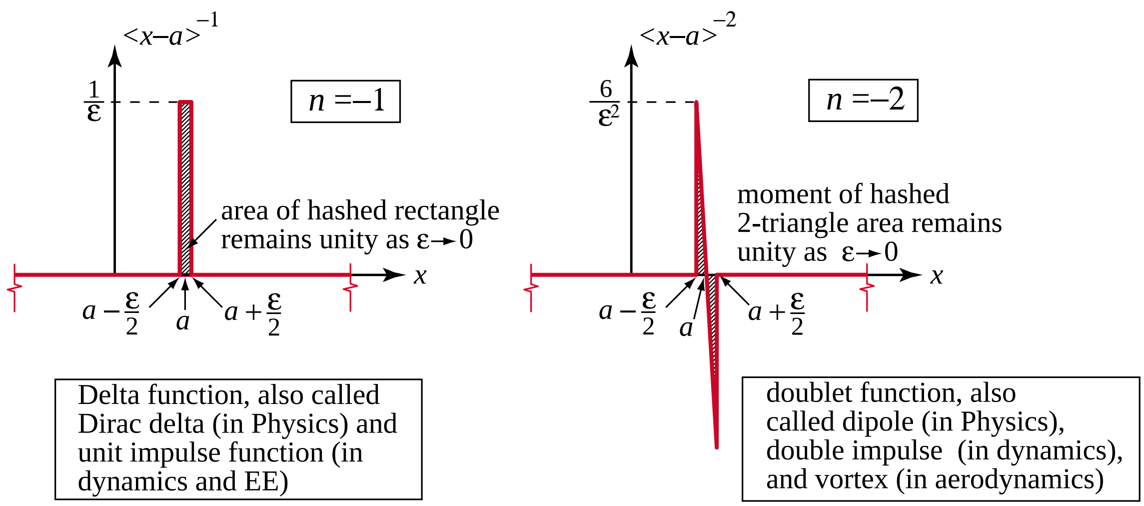


FIGURE 12.3. Discontinuity Functions for $n = -1$ (Delta function, also called Dirac Delta) and $n = -2$ (doublet). Also called *singularity functions*, *distributions* and *generalized functions* in the mathematical literature. (The last name is used in Russian publications.) They do not represent ordinary functions, and as such cannot be conventionally plotted. They are defined only through limit processes, as illustrated in the figure.

§12.1.1. Nonsingular D.F.

Also called *ordinary*. For these n is nonnegative, that is, $n \geq 0$. These may be graphed as conventional functions, as illustrated in Figure 12.2. In this case $\langle x - a \rangle^n$ can be directly defined as $(x - a)^n$ if $x \geq a$, else 0. Consequently n may be interpreted as an exponent. For $n = 0$ the function receives several names noted in Figure 12.2.

§12.1.2. Singular D.F.

If n is a negative integer, $\langle x - a \rangle^n$ is *not* a conventional function. [In advanced mathematics it is known as a distribution (Western literature) or generalized function (Russian literature).] It can be defined as a limit of a sequence of functions, as illustrated in Figure 12.3.

These D.F. exhibit strong singular behavior at $x = a$, and for that reason they are also called *Singularity Functions*. The case $n = -1$ pertains to the so-called delta function, Dirac delta or

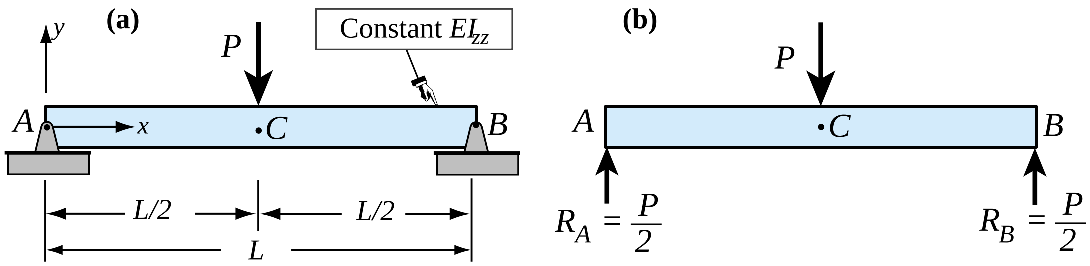


FIGURE 12.4. Beam problem for Example 1: (a) problem definition; (b) FBD to get support reactions.

unit-impulse function, which appears in many fields of engineering and sciences. (For example, collisions in dynamics and particle physics.) Integer n is not an exponent in the usual sense but an index that identifies the singularity strength.

§12.1.3. Application to Beam Deflection Calculations

For beams a point force P (positive up) acting at $x = a$ is mathematically representable as a scaled delta function: $P\langle x-a\rangle^{-1}$. A point moment M_a (positive CW) acting at $x = a$ can be represented by a scaled doublet: $M_a\langle x - a\rangle^{-2}$.

For beam deflection calculations, D.F. are normally used in conjunction with the fourth order method, starting from the applied load $p(x)$. One question often asked is

Should reaction forces be included in $p(x)$?

Answer: that decision is optional. If not included, *they will automatically appear in the integration constants* for the transverse shear force $V_y(x)$ and the bending moment $M_z(x)$.

§12.2. Examples

The following examples illustrate how the method works for three problems of varying complexity.

§12.2.1. Example 1: Simply Supported Beam Under Midspan Point Load

This example problem is defined in Figure 12.4(a). Deflection calculations for this configuration were done in Lecture 11 using the second order method and inter-segment continuity conditions. Here it is solved by the fourth order method and D.F., which foregoes the need for explicit continuity conditions. Write the point load as a delta function of amplitude $-P$:

$$p(x) = -P\langle x - \tfrac{1}{2}L\rangle^{-1}. \tag{12.2}$$

(Note that the reaction forces have not been included in this $p(x)$; as remarked above that decision is optional.) Integrate twice to get transverse shear and bending moment:

$$\begin{aligned} V_y(x) &= \int -p(x)\,dx = P\langle x - \tfrac{1}{2}L\rangle^{0} + C_1, \\ M_z(x) &= \int -V_y(x)\,dx = -P\langle x - \tfrac{1}{2}L\rangle^{1} - C_1\,x + C_2. \end{aligned} \tag{12.3}$$

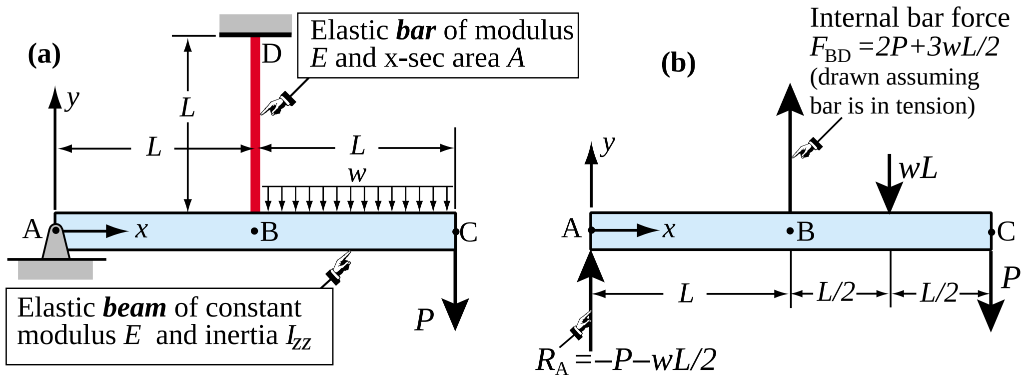


FIGURE 12.5. Structure for Example 2. (a) Problem definition, (b) FBD showing beam reaction R_A and internal bar force F_{BD}. Note that this structure is *statically determinate*. (To visualize this property, try to remove the bar: if so the structure becomes a mechanism and collapses.)

Pause here to apply static BCs. The bending moment $M_z(x)$ is zero at both supports: $M_{zA}(x) = C_2 = 0$ and $M_{zB}(x) = -P(\frac{1}{2}L) - C_1 L = 0 \Rightarrow C_1 = -P/2$. Replace into the moment expression, and integrate twice more:

$$\begin{aligned} M_z(x) &= -P\,\langle x - \tfrac{1}{2}L\rangle^1 + \tfrac{1}{2}P\,x, \\ E\,I_{zz}\,v'(x) &= -\tfrac{1}{2}P\,\langle x - \tfrac{1}{2}L\rangle^2 + \tfrac{1}{4}P\,x^2 + C_3 \\ E\,I_{zz}\,v(x) &= -\tfrac{1}{6}P\,\langle x - \tfrac{1}{2}L\rangle^3 + \tfrac{1}{12}P\,x^3 + C_3\,x + C_4, \end{aligned} \tag{12.4}$$

We now apply the two kinematic BC. The deflection is zero at both supports: $E\,I_{zz}v_A = E\,I_{zz}v(0) = C_4 = 0$ and $E\,I_{zz}v_B = E\,I_{zz}v(L) = -\frac{1}{6}P(\frac{1}{2}L)^3 + \frac{1}{12}P\,L^3 + C_3\,L = 0 \Rightarrow C_3 = -P\,L^2/16$. Replacing into $E\,I_{zz}\,v(x)$ yields the deflection curve, which can be simplified to

$$\boxed{v(x) = -\frac{P}{48E\,I_{xx}}\Big(8\langle x - \tfrac{1}{2}L\rangle^3 - 4\,x^3 + 3L^2x\Big)} \tag{12.5}$$

Evaluating at $x = \frac{1}{2}L$ provides the midspan deflection:

$$\boxed{v_C = v(\tfrac{1}{2}L) = -\frac{P\,L^3}{48E\,I_{zz}}} \tag{12.6}$$

The procedure is faster and less error prone than that used in Lecture 11. On the other hand, it requires the ability to understand and manipulate D.F.

§12.2.2. Example 2: Hinged Beam Propped by Elastic Bar

This is a variant of the second problem discussed in Recitation 5; here it has an additional applied load. It is defined in Figure 12.5(a). Beam AC is simply supported at A and held by an elastic bar BD at halfspan B. Beam AC has constant bending inertia I_{zz}, bar BD has constant cross section A and both members have the same elastic modulus E. The beam is loaded by a point load P at C and a uniform load w over the right halfspan BC. Both P and w are taken as *positive downward*.

In terms of E, A, I_{zz}, L and P, find: (1) axial force F_{BD} in bar BD and reaction at A, (2) deflection v_B at B (this is controlled by the bar elongation) and (3) vertical deflection v_C at C.

To find the axial force in bar BD, do a FBD of the structure taking moments with respect to A so as to get rid of the reaction force. The FBD is pictured in Figure 12.5(b). The moment equilibrium equation (+ CCW): $\sum M_A = F_{BD}\, L - w\, L(3L/2) - P\,(2L) = 0$, yields

$$\boxed{F_{BD} = 2P + \frac{3w\,L}{2}.} \tag{12.7}$$

The reaction at A can be now obtained from y force equilibrium: $\sum F_y = R_A + F_{BD} - w\,L - P = R_A + 2P + 3w\,L/2 - w\,L - P = 0$, which gives

$$\boxed{R_A = -P - \frac{w\,L}{2}.} \tag{12.8}$$

This reaction will act downwards if $P > 0$ and $w > 0$. Under the internal force $F_{BD} = 2P + (3/4)wL$ the bar elongates by δ_{BD}, which is given by the Mechanics of Materials formula

$$\boxed{\delta_{BD} = \frac{F_{BD}\,L}{EA} = \frac{(2P + 3wL/2)\,L}{EA}.} \tag{12.9}$$

This elongation is obviouly equal to the downward beam deflection at B, whence $v_B = v(L) = -\delta_{BD}$ is a kinematic BC. To find the deflection curve we start from the applied forces on the beam. For convenience we take both R_A and F_{BD} as if they were applied loads, and carry them along in compact symbolic form. We write the generic load $p(x)$ using D.F.:

$$\boxed{p(x) = R_A\langle x-0\rangle^{-1} + F_{BD}\langle x-L\rangle^{-1} - w\langle x-L\rangle^{0} - P\langle x-2L\rangle^{-1}.} \tag{12.10}$$

Integrate this twice:

$$\begin{aligned} -V_y(x) &= \int p(x)\,dx = R_A + F_{BD}\langle x{-}L\rangle^0 - w\langle x{-}L\rangle^1 - P\langle x{-}2L\rangle^0 + C_1,\\ M_z(x) &= \int -V_y(x)\,dx = R_A x + F_{BD}\langle x{-}L\rangle^1 - \tfrac{1}{2}w\langle x{-}L\rangle^2 - P\langle x{-}2L\rangle^1 + C_1\,x + C_2. \end{aligned} \tag{12.11}$$

Note that we have replaced $R_A(\langle x-0\rangle^0)$ in the first equation (the expression of the transverse shear force) by $R_A x^0 = R_A$. This is permissible for any DF with *nonnegative* supercript. More generally:

$$\langle x-0\rangle^n \Rightarrow x^n, \quad \text{if } n \geq 0. \tag{12.12}$$

The justification for (12.12) is that x *cannot take negative values* if the coordinate origin is placed at the left end of the beam.

Pause to find C_1 and C_2. The moment at the simple support must be zero: $M_{zA} = M_z(0) = C_2 = 0$ and the transverse shear must equal the negative of the reaction force: $V_{yA} = V_y(0) = -R_A = -R_A + C_1 = 0 \Rightarrow C_1 = 0$. The reason for getting both C_1 and C_2 to vanish is that we incorporated

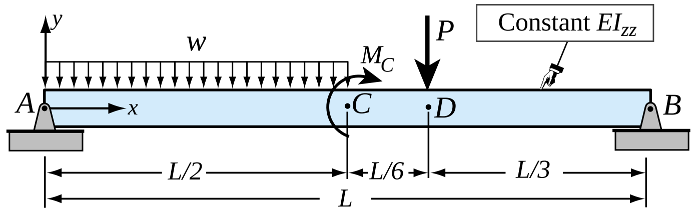


FIGURE 12.6. Beam for Example 3

the beam reactions R_A and F_{BD} from the start in the expression (12.10) for $p(x)$. Two useful checks: $M_{zC} = M_z(2L) = 0$ and $V_{yC} = V_y(2L) = P$.

From now on the P term in the second of (12.11) will be dropped, since for all beam points $x \leq 2L$ and $\langle x - 2L\rangle^n$ vanishes if $n \geq 1$; whence $P\langle x - 2L\rangle^1 = 0$ everywhere. Integrate twice more:

$$\begin{aligned} E I_{zz} v'(x) &= \tfrac{1}{2} R_A x^2 + \tfrac{1}{2} F_{BD}\langle x - L\rangle^2 - \tfrac{1}{6} w\langle x - L\rangle^3 + C_3 \\ E I_{zz} v(x) &= \tfrac{1}{6} R_A x^3 + \tfrac{1}{6} F_{BD}\langle x - L\rangle^3 - \tfrac{1}{24} w\langle x - L\rangle^4 + C_3\, x + C_4. \end{aligned} \tag{12.13}$$

The kinematic BCs are $v_A = v(0) = 0$, because A is a simple support, and $v_B = v(L) = -\delta_{BD} = -F_{BD}\, L/(EA)$, as found above. The first BC gives $C_4 = 0$. The second BC, after some simplifications, yields

$$C_3 = P\left(\frac{L^2}{6} - \frac{2I_{zz}}{A}\right) + \frac{wL}{12}\left(L^2 - \frac{18I_{zz}}{A}\right). \tag{12.14}$$

Replacing C_3, R_A and F_{BD} gives the deflection curve in terms of the data as

$$\begin{aligned} E I_{zz} v(x) &= \tfrac{1}{6}\left(-P - \tfrac{1}{2} wL\right) x^3 + \tfrac{1}{6}\left(2P + \frac{3wL}{2}\right)\langle x - L\rangle^3 \\ &\quad - \tfrac{1}{24} w\langle x - L\rangle^4 - \left(P\left(\frac{L^2}{6} - \frac{2I_{zz}}{A}\right) + \frac{wL}{12}\left(L^2 - \frac{18I_{zz}}{A}\right)\right) x. \end{aligned} \tag{12.15}$$

Evaluating at $x = 2L$ provides the tip deflection:

$$\boxed{v_C = v(2L) = -\frac{PL}{E}\left(\frac{4}{A} + \frac{2L^2}{I_{zz}}\right) - \frac{wL^2}{E}\left(\frac{3}{A} + \frac{7L^2}{24I_{zz}}\right)} \tag{12.16}$$

Note that if $A \to 0$, that is, the bar disappears, the beam deflections go to infinity.

§12.2.3. Example 3: SS Beam With Complicated Loading

The problem is defined in Figure 12.6. The SS beam is subject to three types of applied load: (1) a uniform distributed load w over the left midspan, (2) a point force P at $x = 2L/3$, and a point moment M_C at midspan $x = L/2$. w, P and M_C are positive if acting as shown. The calculation of deflections will be done with the fourth order method in conjunction with D.F. Reactions are not included in the load $p(x)$: they will appear through the integration constants.

The applied load is

$$\boxed{\begin{aligned} p(x) &= -w\langle x-0\rangle^0 + w\langle x-\frac{L}{2}\rangle^0 + P\langle x-\frac{2L}{3}\rangle^{-1} + M_C\langle x-\frac{L}{2}\rangle^{-2} \\ &= -w + w\langle x-\frac{L}{2}\rangle^0 - P\langle x-\frac{2L}{3}\rangle^{-1} + M_C\langle x-\frac{L}{2}\rangle^{-2} \end{aligned}} \tag{12.17}$$

(Note that the load term due to the point moment M_C is positive if this couple acts clockwise. See Vable, p. 492, for a discussion of that topic.) Integrate twice:

$$\begin{aligned} -V_y(x) &= -w\,x + w\langle x-\frac{L}{2}\rangle^1 - P\langle x-\frac{2L}{3}\rangle^0 + M_C\langle x-\frac{L}{2}\rangle^{-1} + C_1, \\ M_z(x) &= -\tfrac{1}{2}w\,x^2 + \tfrac{1}{2}w\langle x-\frac{L}{2}\rangle^2 - P\langle x-\frac{2L}{3}\rangle^1 + M_C\langle x-\frac{L}{2}\rangle^0 + C_1\,x + C_2. \end{aligned} \tag{12.18}$$

The moment at the left simple support must vanish, thus $M_{zA} = M_z(0) = C_2 = 0$ and we dont need to carry C_2 further. But the expression of C_1 from $M_{zB} = M_z(L) = 0$ is involved in terms of the data, so we leave it as is for now. Integrating twice more:

$$\begin{aligned} E I_{zz} v'(x) &= -\tfrac{1}{6}w\,x^3 + \tfrac{1}{6}w\langle x-\frac{L}{2}\rangle^3 - \tfrac{1}{2}P\langle x-\frac{2L}{3}\rangle^2 + M_C\langle x-\frac{L}{2}\rangle^1 + \tfrac{1}{2}C_1\,x^2 + C_3, \\ E I_{zz} v(x) &= -\tfrac{1}{24}w\,x^4 + \tfrac{1}{24}w\langle x-\frac{L}{2}\rangle^4 - \tfrac{1}{6}P\langle x-\frac{2L}{3}\rangle^3 + \tfrac{1}{2}M_C\langle x-\frac{L}{2}\rangle^2 + \tfrac{1}{6}\,C_1\,x^3 + C_3\,x + C_4. \end{aligned} \tag{12.19}$$

The deflection at A must be zero, which immediately gives $C_4 = 0$. The other two integration constants are more complicated functions of the data. Setting $M_{zB} = M_z(L) = 0$ and then $E I_{zz}\, v_B = E I_{zz}\, v(L) = 0$ yields

$$C_1 = \frac{9w - 8P - 24M_C}{24}, \qquad C_3 = \frac{-243w - 512P + 432M_C}{109368} \tag{12.20}$$

Substitution into $E I_{zz}\, v(x)$ produces the deflection curve

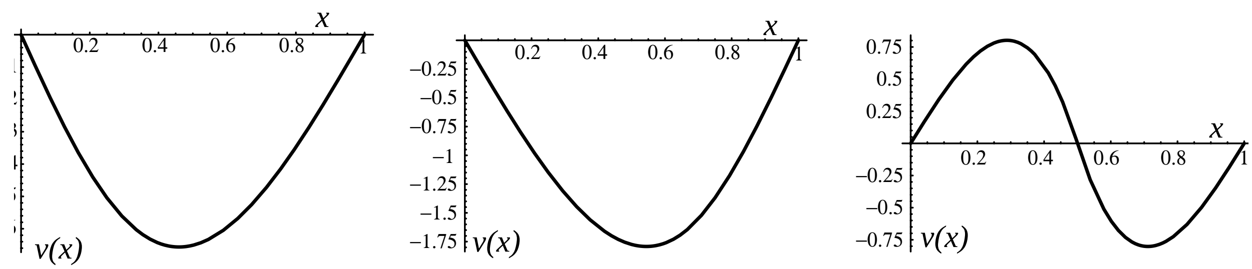
$$\boxed{\begin{aligned} E I_{zz} v(x) = &-\tfrac{1}{24}w\,x^4 + \tfrac{1}{24}w\langle x-\frac{L}{2}\rangle^4 - \tfrac{1}{6}P\langle x-\frac{2L}{3}\rangle^3 + \tfrac{1}{2}M_C\langle x-\frac{L}{2}\rangle^2 \\ &+ \frac{9w - 8P - 24M_C}{144}\,x^3 - \frac{243w - 512P - 432M_C}{109368}x \end{aligned}} \tag{12.21}$$

Figure 12.7 plots deflection curves for three individual load cases.

Evaluating at sections C ($x = \frac{1}{2}L$) and D ($x = \frac{2}{3}L$) gives

$$\boxed{v_C = -\frac{1}{20736\,E I_{zz}}(135\,w - 368\,P), \quad v_D = -\frac{1}{31104\,E I_{zz}}(165\,w - 512\,P + 240\,M_C).} \tag{12.22}$$

The midpsan deflection v_C is not affected by the point moment M_C since this action produces an antisymmetric deflection curve; see Figure 12.7(c).



(a) Deflection curve for w=100, P=0, M_C =0, EI_{zz} = 1 and L=1

(b) Deflection curve for w=0, P=100, M_C =0, EI_{zz} = 1 and L=1

(c) Deflection curve for w=0, P=0, M_C =100, EI_{zz} = 1 and L=1

FIGURE 12.7. Deflection curves for Example 3 beam for three individual load cases.

The *Mathematica* program that solves this problem and produces the plots of Figure 12.7 is shown below.

```
ClearAll[x,w,P,MC,C1,C2,C3,C4]; C2=C4=0;
p=-w+w*UnitStep[x-1/2]-P*DiracDelta[x-2/3]; Print["p(x)=",p];
VVy=Integrate[p,x]+C1;  Print["VVy=",VVy//InputForm];
Vy=-w*x+w*(x-1/2)*UnitStep[x-1/2]-P*UnitStep[x-2/3]+MC*DiracDelta[x-1/2]+C1;
Mz=Integrate[Vy,x]+C2; Mz=Simplify[Mz]; Print["Mz(x)=",Mz];
EIvp=Integrate[Mz,x]+C3; EIvp=Simplify[EIvp]; Print["EIv'(x)=",EIvp];
EIv=Integrate[EIvp,x]+C4;EIv=Simplify[EIv]; Print["EIv(x)=",EIv];
MzA=Mz/.x->0; MzB=Mz/.x->1; EIvA=EIv/.x->0; EIvB=EIv/.x->1;
{MzA,MzB,EIvA,EIvB}=Simplify[{MzA,MzB,EIvA,EIvB}];
Print["MzA=",MzA,"  MzB=",MzB,"  EIvA=",EIvA,"  EIvB=",EIvB];
solC=Simplify[Solve[{MzA==0,MzB==0,EIvA==0,EIvB==0},{C1,C3}]];
Print[solC]; {C1,C3}={C1,C3}/.solC[[1]];
Print["C1=",Together[C1]," C3=",Together[C3]];
EIvx=Simplify[EIv/.solC[[1]]]; Print["EI v(x)=",EIvx];
vC=Simplify[EIvx/.x->1/2]; Print["EI vC=",Together[vC]];
vD=Simplify[EIvx/.x->2/3]; Print["EI vD=",Together[vD]];
vB=Simplify[EIvx/.x->1]; Print["EI vB=",vB];
Plot[100*EIvx/.{w->1,P->0,MC->0},{x,0,1}];
Plot[100*EIvx/.{w->0,P->1,MC->0},{x,0,1}];
Plot[100*EIvx/.{w->0,P->0,MC->1},{x,0,1}];
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