Learning Theory of Optimal Decision Making

Part III: Online learning in adversarial environments

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with thanks to: RLAI group, SZTAKI group

Jean-Yves Audibert, Remi Munos

OUTLINE

1. HIGH LEVEL OVERVIEW OF THE TALKS
2. MOTIVATION
   - What is it?
   - Why should we care?
   - Halving: Find the perfect expert! (0/1 loss)
   - No perfect expert? (0/1 loss)
   - Predicting Continuous Outcomes
3. DISCRETE PREDICTION PROBLEMS
   - Randomized forecasters
   - Weighted Average Forecaster
   - Follow the perturbed leader
4. TRACKING THE BEST EXPERT
   - Fixed share forecaster
   - Variable-share forecaster
   - Other large classes of experts
5. NON-STOCHASTIC BANDIT PROBLEMS
   - Exp3.P: An algorithm for non-stochastic bandit problems
6. CONCLUSIONS

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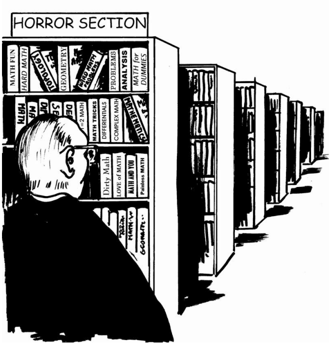
RL
AI

HIGH LEVEL OVERVIEW OF THE TALKS

- Day 1: Online learning in stochastic environments
- Day 2: Batch learning in Markovian Decision Processes
- Day 3: **Online learning in adversarial environments**



Outline

What is it?

Concepts: Agent, Environment, sensations, actions, rewards
Time: $t = 1, 2, \ldots$

Protocol of learning

1. Agent senses x_t coming from Environment
2. Agent sends prediction $\hat{p}_t$ to Environment
3. Environment generates outcome y_t
4. Agent receives loss $\ell_t = \ell(\hat{p}_t, y_t)$ from Environment
5. $t = t + 1$, go to Step 1

Goal: $\sum_{t=1}^{T} \ell_t \to \min$

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OUTLINE

Why should we care?

- No assumptions about the Environment!
- We compare the return with that of algorithms from a set: experts

  "Competitive analysis"
- Results hold for any sequence of observations and returns
- Broader applicability
- Lesson:
  - stochastic, stationary assumptions are not essential for learning
  - algorithms are obtained by robustifying familiar algorithms (plus, some new ideas)

PROTOCOL

Initialization: Algorithm gets N and loss function $\ell(\cdot,\cdot)$

$t := 1$

Main loop:

1. Experts' predictions $f_{1,t}, \ldots, f_{N,t}$ are revealed to Learner
2. Learner computes prediction $\hat{p}_t$
3. Environment computes outcome y_t, which is revealed to Learner
4. Learner learns
5. $t := t + 1$; go to Step 1

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Notation

- (Total) loss of expert i:

$$L_{i,n} = \sum_{t=1}^{n} \ell(f_{it}, y_t)$$

- (Total) loss of best expert:

$$L_n^* = \min_i L_{in}$$

- (Total) loss of algorithm:

$$\hat{L}_n = \sum_{t=1}^{n} \ell(\hat{p}_t, y_t)$$

- (Total) regret:

$$R_n = \hat{L}_n - L_n^*$$

- Goal: Design algorithm that keeps the regret small

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When there is a infallible expert..

- Binary world:
$$\mathcal{Y} = \mathcal{D} = \{0, 1\}$$
- Loss:
$$\ell(p, y) = \mathbb{I}_{\{p \neq y\}}$$
- N experts
- Expert predictions: $f_{i1}, f_{i2}, \ldots \in \{0, 1\}$

Assumption

There is an expert that never makes a mistake.

Problem

How to keep the regret small?

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Problem

How to keep the regret small?

Halving Algorithm

- Keep regret small ⇒ Learn from mistakes
- Idea:
  - Eliminate immediately experts that make a mistake
  - Take majority vote of remaining experts

⇒ "Halving Algorithm"

[Barzdin and Freivalds, 1972, Angluin, 1988]

Theorem (Finite regret for the Halving algorithm)

No matter what $y_1, y_2, \ldots$ is,

$$R_n = \hat{L}_n - L_n^* \leq \lfloor \log_2 N \rfloor .$$

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- Weight $w_{it} \in \{0, 1\}$:
  Is expert i alive at time t? (after y_t is received)
- Let $w_{i0} = 1$, $i = 1, 2, \ldots, N$.
- $W_t = \sum_{i=1}^{N} w_{it}$: Number of alive experts at time t
- $\hat{L}_t$: number of mistakes up to time t (including time t)

CLAIM

If Halving makes a mistake ($\ell(\hat{p}_t, y_t) = 1$) then $W_t \leq W_{t-1}/2$.
Further W_t cannot grow.

COROLLARY

$W_t \leq W_0/2^{\hat{L}_t} = N/2^{\hat{L}_t}$.

Finish: Now, $1 \leq W_t$, hence $1 \leq N/2^{\hat{L}_t}$, i.e., $\hat{L}_t \leq \lfloor \log_2 N \rfloor$.

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No Perfect Expert: "Weighted Majority"

- Problem with elimination: fails if there is no perfect expert!
- Improved algorithm: "Weighted Majority"
  [Littlestone and Warmuth, 1994]
  - Keep weights positive!
  - Have weights of mistaken experts decay:

$$w_{it} = \beta w_{i,t-1}, \text{ if } f_{it} \neq y_t \; (0 < \beta < 1)$$

  - Keep majority vote!

Theorem (Loss bound for WM)

$$\hat{L}_n \leq \left\lfloor \frac{\log_2(\frac{1}{\beta}) L_n^* + \log_2 N}{\log_2(\frac{2}{1+\beta})} \right\rfloor .$$

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Outline

Predicting Continuous Outcomes

- What if $\mathcal{Y} = \mathcal{D} = [0,1]$ or $\mathbb{R}^d$ (or a convex subset of some vector space)?
- Bounded loss: $\ell : \mathcal{D} \times \mathcal{Y} \to [0,1]$
- Example: $\mathcal{D} = \mathcal{Y} = [0,1]$, $\ell(p,y) = \frac{1}{2}|p-y|$.
- Can we generalize the previous algorithm?
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LOSS BOUND

THEOREM (LOSS BOUND FOR THE EWA FORECASTER)

Assume that $\mathcal{D}$ is a convex subset of some vector-space. Let $\ell : \mathcal{D} \times \mathcal{Y} \to [0, 1]$ be convex in its first argument and consider the loss $\hat{L}_n$ of EWA. Then:

$$\hat{L}_n \le L_n^* + \frac{\ln N}{\eta} + \frac{\eta}{8} n.$$

With $\eta = \sqrt{\frac{8 \ln N}{n}}$,

$$\hat{L}_n \le L_n^* + \sqrt{\frac{n \ln N}{2}}.$$

ADAPTIVE AND SELF-CONFIDENT FORECASTERS

- Problem: η depends on n, the horizon
- Small losses
  - Loss bound for WM, 0/1-predictions:

$$\hat{L}_n \le \left\lfloor \frac{\log_2(\frac{1}{\beta})L_n^* + \log_2 N}{\log_2(\frac{2}{1+\beta})} \right\rfloor.$$

  - If $L_{in} = 0$ for some expert then the regret is finite!
  - Regret bound for EWA:

$$\hat{L}_n \le L_n^* + \sqrt{n/2 \, \ln N} \xrightarrow{n\to\infty} \infty \text{ even if } L_n^* = 0!$$

THEOREM ([AUER ET AL., 2002B])

Consider EWA with $\eta_t = c\sqrt{\ln N / L_{t-1}^*}$, $c > 0$. Under the same conditions as in the previous theorem for some $\kappa > 0$,

$$R_n \le 2\sqrt{2L_n^* \ln N} + \kappa \ln N.$$

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Binary prediction problems

- Binary prediction problem:

$$\mathcal{D} = \mathcal{Y} = \{0, 1\}, \quad \ell(p, y) = \mathbb{I}_{\{p \neq y\}}$$

- Bound of WM:

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- Question: Can we have an additive bound, like that of EWA:

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with $B(n, N) = o(n)$?
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Proposition

Consider binary prediction problems and pick any deterministic forecaster. Let $\hat{L}_n(y_{1:n})$ be the forecaster's loss on $y_{1:n}$. Then $\exists y_{1:n}$ s.t. $\hat{L}_n(y_{1:n}) = n$.

Proof.

Induction on n. □

Corollary

No deterministic forecaster can have sublinear regret.

Proof.

Let $N = 2$, $f_{1t} \equiv 0$, $f_{2t} \equiv 1$. Then $\forall\, y_{1:n}$, $L_n^*(y_{1:n}) \le n/2$.
Pick some $y_{1:n}$ that forces $\hat{L}_n(y_{1:n}) = n$. □

Idea

Randomize the forecaster!

WHY RANDOMIZE?

PROPOSITION

Consider binary prediction problems and pick any deterministic forecaster. Let $\hat{L}_n(y_{1:n})$ be the forecaster's loss on $y_{1:n}$. Then $\exists y_{1:n}$ s.t. $\hat{L}_n(y_{1:n}) = n$.

PROOF.

Induction on n. □

COROLLARY

No deterministic forecaster can have sublinear regret.

PROOF.

Let $N = 2$, $f_{1t} \equiv 0$, $f_{2t} \equiv 1$. Then $\forall\, y_{1:n}$, $L_n^*(y_{1:n}) \leq n/2$. Pick some $y_{1:n}$ that forces $\hat{L}_n(y_{1:n}) = n$. □

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Outline

RANDOMIZED FORECASTERS

- Can we use EWA to get sublinear regret?
  .. but predictions must be binary!
- Crucial differences:
  - predictions cannot be combined
  - $\ell(\cdot, y)$ is not convex
- Idea: "Simulate EWA":

$$I_t \sim (w_{1,t-1}, \ldots, w_{N,t-1}), \hat{p}_t = f_{I_t,t}.$$

PROTOCOL

Initialization: Algorithm gets N and ε, $t := 1$
At time t

1. Experts' predictions $f_{1,t}, \ldots, f_{N,t}$ are revealed to Learner
2. Learner computes I_t
3. Environment computes outcome Y_t
4. Losses $\ell(1, Y_t), \ell(2, Y_t), \ldots, \ell(N, Y_t)$ is revealed to Learner

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OUTLINE

Weighted Average Forecaster [Littlestone and Warmuth, 1994]

Previous result on EWA:

Theorem (Loss bound for the EWA forecaster)

Assume that $\mathcal{D}$ is a convex subset of some vector-space. Let $\ell : \mathcal{D} \times \mathcal{Y} \to [0,1]$ be convex in its first argument. Then, for EWA ($\hat{p}_t = \frac{\sum_i w_{i,t-1} f_{it}}{\sum_j w_{j,t-1}}$, $w_{i,t-1} = e^{-\eta L_{i,t-1}}$) it holds:

$$\hat{L}_n - L_n^* \leq \frac{\ln N}{\eta} + \frac{\eta}{8} n.$$

With $\eta = \sqrt{\frac{8 \ln N}{n}}$, $\hat{L}_n - L_n^* \leq \sqrt{n/2 \; \ln N}$.

- Let $f_{it} = e_i$ (ith unit vector), $\hat{p}_{it} = \frac{w_{i,t-1}}{\sum_{j=1}^N w_{j,t-1}}$
- $\bar{\ell}(p,y) \stackrel{\text{def}}{=} \sum_{i=1}^N p_i \ell(i,y)$, $\Rightarrow \bar{\ell}$ is convex in p
- $\mathcal{D} = \Delta_1 \stackrel{\text{def}}{=} \{ p \in \mathbb{R}^N \,|\, p_i \geq 0, \sum_j p_i = 1 \} \subset \mathbb{R}^N$ is convex.

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BOUND ON THE PSEUDO-EXPECTED REGRET

EWA: $\hat{p}_t = \frac{\sum_i w_{i,t-1} f_{it}}{\sum_j w_{j,t-1}}$, $w_{i,t-1} = e^{-\eta L_{i,t-1}}$

THEOREM (LOSS BOUND FOR THE EWA FORECASTER: RANDOMIZED PREDICTIONS)

Let $\ell : \underline{N} \times \mathcal{Y} \to [0,1]$. Then, for EWA it holds:

$$\overline{L}_n - L_n^* \leq \frac{\ln N}{\eta} + \frac{\eta}{8} n.$$

With $\eta = \sqrt{\frac{8 \ln N}{n}}$, $\overline{L}_n - L_n^* \leq \sqrt{n/2\ \ln N}$. Here

$$\overline{L}_n = \sum_{t=1}^{n} \overline{\ell}(\hat{p}_t, Y_t) = \sum_{t=1}^{n} \sum_{i=1}^{N} \hat{p}_{it} \ell(i, Y_t).$$

Note:

$$\overline{\ell}(\hat{p}_t, Y_t) = \mathbb{E}\left[\ell(I_t, Y_t) \mid Y_{1:t}, I_{1:t-1}\right] (= \mathbb{E}_t\left[\ell(I_t, Y_t)\right]).$$

Bound on the pseudo-expected regret

EWA: $\hat{p}_t = \frac{\sum_i w_{i,t-1} f_{it}}{\sum_j w_{j,t-1}}$, $w_{i,t-1} = e^{-\eta L_{i,t-1}}$

Theorem (Loss bound for the EWA forecaster: Randomized predictions)

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Bound on the Actual Regret

- What about $\hat{L}_n - L_n^*$??
- $\hat{L}_n = \sum_{t=1}^{n} \ell(I_t, Y_t) \overset{??}{\approx} \sum_{t=1}^{n} \overline{\ell}(\hat{p}_t, Y_t) = \overline{L}_n$
- $\overline{\ell}(\hat{p}_t, Y_t)$ is the (conditional) "expected value" of $\ell(I_t, Y_t)$
- Hoeffding $\Rightarrow$ Sums of i.i.d. random variables are $\sqrt{n}$-close to their expectations!

  Extension to martingales $\Rightarrow$ Hoeffding-Azuma

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Bound on the random regret

Theorem (Loss bound for the EWA forecaster: Random regret)

Let $\ell : \underline{N} \times \mathcal{Y} \to [0, 1]$*. Then, for EWA it holds:*

$$\hat{L}_n - L_n^* \leq \frac{\ln N}{\eta} + \frac{\eta}{8} n + \sqrt{\frac{n}{2} \ln(1/\delta)}$$

With $\eta = \sqrt{\frac{8 \ln N}{n}}$,

$$\hat{L}_n - L_n^* \leq \sqrt{\frac{n}{2} \ln N} + \sqrt{\frac{n}{2} \ln(1/\delta)}.$$

Small losses

- Previous "small-loss" bound:

$$2\sqrt{2L_n^* \ln N} + \kappa \ln N$$

- Random fluctuations: add $\sqrt{n/2 \ln(1/\delta)}$ – too big!
- Bernstein's inequality uses the "predictable variance" to bound the fluctuations
- Bound on the "predictable variance":

$$\begin{aligned}\mathbb{E}_t\left[\left(\ell(I_t, Y_t) - \overline{\ell}(\hat{p}_t, Y_t)\right)^2\right] &= \mathbb{E}_t\left[\ell(I_t, Y_t)^2\right] - \overline{\ell}^2(\hat{p}_t, Y_t) \\ &\le \mathbb{E}_t\left[\ell(I_t, Y_t)^2\right] \le \mathbb{E}_t\left[\ell(I_t, Y_t)\right] = \overline{\ell}(\hat{p}_t, Y_t)\end{aligned}$$

- $\Rightarrow$ the effect of random fluctuations is comparable with the bound on the expected regret:

$$\sum_{t=1}^{n}\left(\ell(I_t, Y_t) - \overline{\ell}(\hat{p}_t, Y_t)\right) \le \sqrt{2\overline{L}_n \ln(1/\delta)} + \frac{2\sqrt{2}}{3}\ln(1/\delta).$$

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$$\leq \quad \mathbb{E}_t\left[\ell(I_t, Y_t)^2\right] \leq \mathbb{E}_t\left[\ell(I_t, Y_t)\right] = \bar{\ell}(\hat{p}_t, Y_t)$$

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Outline

Follow the Leader

- Does it work?
- Take $N = 2$:

$$\begin{array}{ll} \ell(1, y_t): & \frac{1}{2}, 0, 1, 0, 1, 0, \ldots \\ \ell(2, y_t): & \frac{1}{2}, 1, 0, 1, 0, 1, \ldots \end{array}$$

- Choices:

$$\begin{array}{ll} \ell(1, y_t): & \frac{1}{2}^{L_{11}=.5}, 0^{L_{12}=.5}, 1^{L_{13}=1.5}, 0^{L_{14}=1.5}, 1^{L_{15}=2.5}, 0, \ldots \\ \ell(2, y_t): & \frac{1}{2}^{L_{21}=.5}, 1^{L_{22}=1.5}, 0^{L_{22}=1.5}, 1^{L_{23}=2.5}, 0^{L_{24}=2.5}, 1, \ldots \end{array}$$

- $\Rightarrow \hat{L}_n = n - 2 + 0.5$, whilst $L_{in} \le n/2$, $i = 1, 2$,

$$\hat{L}_n - L_n^* \ge n/2 - 1.5$$

FOLLOW THE PERTURBED LEADER [HANNAN, 1957]

- Follow the perturbed leader (randomized fictitous play):

$$
\begin{aligned}
I_t &= \operatorname{argmin}_{i=1,\dots,N} \left(L_{i,t-1} + Z_{it} \right), \\
Z_t &\sim f(\cdot), \quad \text{i.i.d.}
\end{aligned}
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- Goal: develop bound on $\overline{L}_n$!
- Relate to BEH:

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FPL BOUND

THEOREM (FPL BOUND [KALAI AND VEMPALA, 2003])

Let $\ell : \underline{N} \times \mathcal{Y} \to [0,1]$ *and consider FPL! Let*

$$Z_t \sim f(\cdot), \quad f(z) = (\tfrac{\eta}{2})^N e^{-\eta \|z\|_1}.$$

Then

$$\mathbb{E}\left[\hat{L}_n\right] \leq e^{\eta}\left(\mathbb{E}\left[L_n^*\right] + \frac{2(1+\ln N)}{\eta}\right).$$

Choose

$$\eta = \min\left\{1, \sqrt{\frac{2(1+\ln N)}{(e-1)L_n^*}}\right\}.$$

Then

$$\mathbb{E}\left[L_n\right] - \mathbb{E}\left[L_n^*\right] \leq 2\sqrt{2L_n^*(e-1)(1+\ln N)} + 2(e+1)(1+\ln N).$$

Tracking the best expert [Herbster and Warmuth, 1998]

- Discrete prediction problem
- Want to compete with 'compound action sets':

$$B_{n,m} = \{(i_1, \ldots, i_n) \mid s(i_1, \ldots, i_n) \leq m\},$$

  where $s(i_1, \ldots, i_n) = \sum_{t=2}^{n} \mathbb{I}_{\{i_{t-1} \neq i_t\}}$ is the number of switches.
- Shorthand notation $i_{1:n} = (i_1, \ldots, i_n)$
- Regret:

$$R_{n,m} \stackrel{\text{def}}{=} \sum_{t=1}^{n} \ell(I_t, y_t) - \min_{i_{1:n} \in B_{n,m}} \sum_{t=1}^{n} \ell(i_t, y_t).$$

- Instead we use $\overline{R}_{n,m}$, where

$$\overline{R}_{n,m} \stackrel{\text{def}}{=} \max_{i_{1:n} \in B_{n,m}} \overline{R}(i_{1:n}),\ \overline{R}(i_{1:n}) \stackrel{\text{def}}{=} \sum_{t=1}^{n} \overline{\ell}(p_t, y_t) - \sum_{t=1}^{n} \ell(i_t, y_t).$$

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Outline

Randomized EWA applied to tracking problems

- Action set: $B_{n,m}$.
- We always select a compound, but just play the next primitive action.
- Previous regret bound gives:

$$\overline{R}_{n,m} \leq \sqrt{\frac{n}{2}\ln(|B_{n,m}|)}.$$

- $M = |B_{n,m}| \leq ?$
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FIXED-SHARE FORECASTER (FSF)

Initialize: $w_{i0} = 1/N$.

1. Draw expert index I_t from $w_{i,t-1}/\sum_{j=1}^{N} w_{j,t-1}$.
2. Send I_t to Environment
3. Receive y_t and losses $(\ell(i, y_t))_i$ from Environment
4. Update weights:
5. $v_{it} := w_{i,t-1} e^{-\eta\ell(i,y_t)}$
6. $V_t := \sum_{j=1}^{N} v_{jt}$
7. $w_{it} := \frac{\alpha}{N} V_t + (1-\alpha) v_{it}$

Regret bound for FSF

Theorem ([Herbster and Warmuth, 1998])

Consider a discrete prediction problem and pick any sequence $y_{1:n}$. *For any compound action* $i_{1:n}$,

$$\overline{R}(i_{1:n}) \leq \frac{s(i_{1:n})+1}{\eta} \ln N + \frac{1}{\eta} \ln \left(\frac{1}{\alpha^{s(i_{1:n})}(1-\alpha)^{n-s(i_{1:n})}} \right) + \frac{\eta}{8} n.$$

For $0 \leq m \leq n$, $\alpha = m/(n-1)$, *with a specific choice of* $\eta = \eta(n, m, N)$,

$$\overline{R}_{n,m} \leq \sqrt{\frac{n}{2} \left((m+1) \ln N + (n-1) H\left(\frac{m}{n-1}\right) + \ln \left(\frac{1}{1-\frac{m}{n-1}} \right) \right)}.$$

OUTLINE

Variable-share forecaster: Algorithm

Variable-share forecaster (VSF)

Initialize: $w_{i0} = 1/N$.

1. Draw primitive action I_t from $w_{i,t-1} / \sum_{j=1}^N w_{j,t-1}$.
2. Observe y_t, losses $\ell(i, y_t)$ (suffers loss $\ell(I_t, y_t)$)
3. Compute $v_{it} = w_{i,t-1} e^{-\eta \ell(i,y_t)}$
4. Let $w_{it} = \frac{1}{N-1} \sum_{j \neq i} \left(1 - (1-\alpha)^{\ell(j,y_t)}\right) v_{jt} + (1-\alpha)^{\ell(i,y_t)} v_{it}$.

   // If loss of current action is small, stay at it, otherwise encourage switching!

- Result: For binary losses, $\frac{n - s(i_{1:n}) - 1}{\eta} \ln \frac{1}{1-\alpha}$ is replaced by $s(i_{1:n}) + \frac{1}{\eta} L(i_{1:n}) \ln \frac{1}{1-\alpha}$.
- Small complexity ($s(i_{1:n})$) and small loss ($L(i_{1:n})$): big win

VARIABLE-SHARE FORECASTER: ALGORITHM

VARIABLE-SHARE FORECASTER (VSF)

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- Small complexity ($s(i_{1:n})$) and small loss ($L(i_{1:n})$): big win

VARIABLE-SHARE FORECASTER (VSF)

Initialize: $w_{i0} = 1/N$.

1. Draw primitive action I_t from $w_{i,t-1} / \sum_{j=1}^{N} w_{j,t-1}$.
2. Observe y_t, losses $\ell(i, y_t)$ (suffers loss $\ell(I_t, y_t)$)
3. Compute $v_{it} = w_{i,t-1} e^{-\eta \ell(i,y_t)}$
4. Let $w_{it} = \frac{1}{N-1} \sum_{j \neq i} \left(1 - (1-\alpha)^{\ell(j,y_t)}\right) v_{jt} + (1-\alpha)^{\ell(i,y_t)} v_{it}$.

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OTHER EXAMPLES

- Tree experts (side info); e.g. [D.P. Helmbold, 1997]
- Shortest path FPL: [Kalai and Vempala, 2003]; additive losses
- Shortest path EWA [György et al., 2005]; compression – best scalar quantizers [György et al., 2004]
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- Further applications:
  - Sequential allocation
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Bandit setting

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Feedback is restricted to the expert (action) chosen

Protocol

Initialization: Algorithm gets N and ℓ, $t := 1$

At time t:

1. Expert predictions are revealed to Learner
2. Learner chooses expert $I_t \in \{1, \ldots, N\}$
3. Environment generates outcome Y_t
4. Learner receives $\ell_t = \ell(I_t, Y_t)$ from Environment
5. $t := t + 1$; go to Step 1

No feedback is received for actions $i \neq I_t$

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OBSERVATION

That we do not receive feedback for all experts does not mean that no "appropriate" feedback can be derived for them!

- Consider randomized EWA and expected losses
- Only $\mathbb{E}\left[\ell(i, Y_t)\right]$ matters:
  When ℓ, ℓ' are such that for $\forall i, t$: $\mathbb{E}\left[\ell(i, Y_t)\right] = \mathbb{E}\left[\ell'(i, Y_t)\right]$ and $0 \le \ell, \ell' \le 1$, then the bounds on the expected regret of EWA are the same for both ℓ and ℓ'.
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OUTLINE

Feedback for all experts!

- Work with gains: $g(i, Y_t) = 1 - \ell(i, Y_t)$
- Proposed feedback:

$$\tilde{g}(i, Y_t) = \begin{cases} \frac{g(i,Y_t)}{p_{i,t}}, & \text{if } I_t = i \\ 0 & \text{otherwise.} \end{cases}$$

- Compact notation: $\tilde{g}(i, Y_t) = \mathbb{I}_{\{I_t = i\}} g(I_t, Y_t)/p_{I_t,t}$.
- Prop: If $p_{j,t} > 0$ holds $\forall j \in \underline{N}$, where p_{jt} depends on $g(I_1, Y_1), \ldots, g(I_{t-1}, Y_{t-1})$, then $\forall i \in \underline{N}$,

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Exp3.P($\eta, \beta, \gamma > 0$) [Auer et al., 2002a]

Initialize: $w_{i0} = 1$, $p_{i1} = 1/N$

A time t do:

1. Compute action selection probabilities:

$$p_{it} = (1 - \gamma) \frac{w_{i,t-1}}{\sum_{j=1}^{N} w_{j,t-1}} + \gamma \frac{1}{N}.$$

2. Select $I_t \sim p_{.,t}$
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$$g'(i, Y_t) = \tilde{g}(i, Y_t) + \frac{\beta}{p_{it}}.$$

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Regret bound for Exp3.P

Theorem (Regret of Exp3.P [Auer et al., 2002a])

Consider Exp3.P. Let $0 < \delta < 1$ *arbitrary,* $n \geq 8N \ln(N/\delta)$,

$$\gamma \leq \frac{1}{2}, \quad 0 < \eta \leq \frac{\gamma}{2N}, \quad \sqrt{\frac{1}{nN} \ln \frac{N}{\delta}} \leq \beta \leq 1.$$

Then with probability at least $1 - \delta$*, we have*

$$\hat{L}_n - L_n^* \leq n(\gamma + \eta(1+\beta)N) + \frac{\ln N}{\eta} + 2nN\beta.$$

Choosing β *as its lower bound,* η *as its upper bound,* $\gamma = 4N\beta/(3+\beta)$*, then*

$$\hat{L}_n - \min_i L_{in} \leq \frac{11}{2}\sqrt{nN \ln \frac{N}{\delta}} + \frac{\ln N}{2}.$$

Note: $n \geq 8N \ln(N/\delta)$ ensures that γ (2nd part) is at most $1/2$.

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Why use inflated values?

- $\beta = 0 \Rightarrow$ Exp3
- The expected regret of Exp3 is

$$O(\sqrt{nN \ln N}).$$

- Problem:

  No high-probability bound on the actual regret!

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Losses vs. Gains

- The algorithm could work with losses, too!
- Gains:
  - When an action becomes bad, its weight ceases to grow
  - When an action becomes good, its weight grows
- Losses:
  - When an action becomes bad, its weight decreases
  - When an action becomes good (loss=0), its weight is not decreased
- Working with losses:
  - Better when an action becomes bad
  - Quickly reacts then
- Working with gains:
  - Better when an action becomes good
  - Warning: Takes N/γ steps to find out about this action!
- Work with losses: [Cesa-Bianchi et al., 2005]

Losses vs. Gains

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Lower bound

Theorem (Minimax lower bound [Auer et al., 2002a])

Fix $n, N \geq 1$. Let $n > N/(4\ln(4/3))$ and assume that the output space $\mathcal{Y}$ has at least 2^N elements. Then there exists a loss function such that

$$\sup_{y_{1:n}} \left(\mathbb{E}\left[\hat{L}_n\right] - \min_{i=1,\ldots,N} L_{in} \right) \geq \frac{\sqrt{2}-1}{\sqrt{23\ln(4/3)}} \sqrt{nN}.$$

Proof.

- One uniform random variable decides which action should be the best.
- Payoffs are Bernoulli $(1/2, 1/2)$, except for the best arm, which is Bernoulli $(1/2-\epsilon, 1/2+\epsilon)$.
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    - Ridge regression forecaster [Vovk, 2001, Azoury and Warmuth, 2001]
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    - ...
- Great book: [Cesa-Bianchi and Lugosi, 2006]

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