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Generation and propagation of a q-deformed type of $d^N \neq 0$ curvature

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Abstract

We present an expression for curvature with q-deformed calculus such as considered in [1, 2, 3]. By exploiting the persistence of Bianchi's second identity, we suggest a way to attach physical meaning to the q parameters and $d^N \neq 0$ condition by introducing a physical current, an example of which may be obtained by a procedure outlined in [4].

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I. INTRODUCTION

In this section we explain notation used in the following sections. The appearance of the "constant" deformation parameter q is only formal since its index structure in a coordinate basis $\{dx^{\mu}\}$ of differential 1-forms can be involved, where general indices will be denoted by $i, j, k, l, ...$ meanwhile spacetime indices will be denoted by $\mu, \nu, \alpha, \beta, ...$ In order to illustrate this point, we will show the explicit form of the q-symmetrization in the case of the 2-form of the electromagnetic field in section IV. $\mathcal{M}$ will be an n-dimensional differentiable manifold and $T(\mathcal{M})$ will be the tangent bundle over $\mathcal{M}$. The differential forms will be regarded as belonging to the algebra Λ_q of sections of the q-symmetrized tensor algebra $\mathcal{T}^*(\mathcal{M})$ of the linear functionals $T^*(\mathcal{M}) = \{u : T(\mathcal{M}) \to \mathbb{C}\}$ on $T(\mathcal{M})$. That is,

$$\begin{aligned}
&\mathcal{T}^*(\mathcal{M}) = \mathbb{C} \oplus T^*(\mathcal{M}) \oplus T^*(\mathcal{M})^{2\otimes_q} \oplus ... \oplus T^*(\mathcal{M})^{n\otimes_q},\\
&T^*(\mathcal{M})^{k\otimes_q} = T^*(\mathcal{M}) \otimes_q T^*(\mathcal{M})^{(k-1)\otimes_q},\\
&\Lambda_q = \{f : \mathcal{M} \to \mathcal{T}^*(\mathcal{M})\} = \Lambda_q^0 \oplus \Lambda_q^1 \oplus \Lambda_q^2 \oplus ... \oplus \Lambda_q^n,\\
&\Lambda_q^k = \{f : \mathcal{M} \to T^*(\mathcal{M})^{k\otimes_q}\},\\
&\Lambda_q^0 = \mathcal{F}(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{C}\}.
\end{aligned} \tag{1}$$

Differentiation is from the left and whenever the q-deformed exterior differential d passes over a q-deformed j-form from the left, a factor of q^j is piked up. That is if $f \in \Lambda_q^{|f|}$, $g \in \Lambda_q^{|g|}$ then

$$d : \Lambda_q \to \Lambda_q, \quad \Lambda_q^k \to \Lambda_q^{(k+|d|)\ mod\ n}, \quad d(fg) = df\ g + q^{|f||d|} f\ dg, \tag{2}$$

where we take $|d| = 1$. ω will denote a 2-array (or matrix) of q-deformed differential 1-forms.

We also adopt the following index summation convention which is most important for section IV. An index is summed over only if it appears more than once (twice, thrice etc) on one side of an equation but does not appear on the opposite side of the equation. For example, the indices i and j in $a_{ij} = u_i v_{ij}$ are not summed over. Similarly, i, j are not summed over but k, l are summed over in $a_{ij} = q_{kj} b_{kl} c_{lik} d_j$. There are exception which should be clear from context. Examples of such exceptions include the assignment of values to the components of an array, eg. in $a_{ij} = 2 \ \ \forall i, j$, and in the notation for a matrix $M = (M_{ij}) = (\lambda_i \alpha_{ij})$ or a vector $v = (v_i) = (\lambda_i u_i)$ or any other array.

II. CONNECTION AND CURVATURE

Consider a tangent vector field $v = \tilde{v}^i \partial_i : \mathcal{M} \to T(\mathcal{M})$ in the coordinate basis $\{\partial_i\}$ with $d\partial_i = \omega_i{}^j \partial_j$ (or simply $d\partial = \omega\partial$) and $\omega = (\tilde{\omega}_i{}^j{}_\alpha dx^\alpha)$ being a 1-form valued matrix or a connection 1-form. Then one finds that

$$\begin{aligned}
&d^k v = d^k(\tilde{v}\partial) = \sum_{r=0}^{k} C_q^{(k,r)} d^{k-r}\tilde{v}\ \Omega_r\ \partial \equiv \sum_{r=0}^{k} C_q^{(k,r)} d^{k-r}\tilde{v}\ d^r\partial \quad \in\ \Lambda_q^k \otimes T(\mathcal{M}),\\
&C_q^{(k,r)} = \frac{[k]_q!}{[r]_q![k-r]_q!},\\
&[n]_q! = [n]_q..[3]_q[2]_q[1]_q,\\
&[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + .. + q^{n-2} + q^{n-1},
\end{aligned} \tag{3}$$

where the sequence of curvature forms (Ω_r) is given recursively by

$$
\begin{aligned}
&\Omega_0 = 1,\\
&\Omega_1 = \omega,\\
&\Omega_2 = d\omega + q\omega^2,\\
&\Omega_3 = d\Omega_2 + q^2\Omega_2\omega,\\
&....\\
&\Omega_k = d\Omega_{k-1} + q^{k-1}\Omega_{k-1}\omega.
\end{aligned} \tag{4}
$$

Remarks:

- $\Omega_k = (\Omega_k)_i{}^j \in \Lambda_q^k \ \ \forall i, j$ and Ω_k does not exist in dimension $n < k$.
- If for a particular k we set $d^k = 0,\ \Omega_k = 0$ then a solution is given by a Maurer-Cartan form for $\mathcal{M} = GL(N)$ thus:

$$
\begin{aligned}
&\Theta_{k-1} = d^{k-1}g\ g^{-1}, \quad g \in GL(N),\\
&\Theta_1 = \theta = dg\ g^{-1},\\
&\Theta_r = \begin{cases} d\Theta_{r-1} + q^{r-1}\Theta_{r-1}\ \theta, & 0 \leq r < k \\ 0, & r \geq k. \end{cases}
\end{aligned} \tag{5}
$$

- On the other hand, $d\Omega_k = 0,\quad q^k = 1 \quad \Rightarrow \qquad \Omega_{k+r} = \Omega_k\ \Omega_r, \qquad \Omega_{r_1 k + r_2} = \Omega_k{}^{r_1}\ \Omega_{r_2}$, where r, r_1, r_2 as well as k are positive integers.
- Finally for the case we are most concerned with, it can be checked that $d^k = 0,\quad [k]_q = 0$ implies Bianchi' 2nd identity

$$
D\Omega_k = d\Omega_k - [\omega, \Omega_k] = d\Omega_k - \omega\Omega_k + \Omega_k\omega = 0, \tag{6}
$$

$$
\begin{aligned}
\Rightarrow \quad & d\ \mathrm{Tr}\Omega_k + (1 - q^k)\mathrm{Tr}(\Omega_k\omega) = d\ \mathrm{Tr}\Omega_k = 0,\\
& \mathrm{Tr}(\omega\Omega_k) = q^k\mathrm{Tr}(\Omega_k\omega).
\end{aligned} \tag{7}
$$

  Therefore, locally in $\mathcal{M}$, one may be able to write $\mathrm{Tr}\Omega_k = d^{k-1}U_1$; the "constant" numbers $z_k = \int \mathrm{Tr}\Omega_k$ may be used to characterize topological nontriviality on $\mathcal{M}$.

  We also remark here that (6) has a solution of the form

$$
\Omega_k = \mathcal{P}e^{\int \omega}\ C_k\ \bar{\mathcal{P}}e^{-\int \omega}, \quad dC_k = 0 \tag{8}
$$

  and that a Lagrange multiplier type analytic continuation can be carried out as explained in [4]. We will now consider the following alternative method of analytic continuation.

III. THE SOURCE EQUATION

Since $d^k = 0,\quad [k]_q = 0$ implies

$$
d\Omega_k - \omega\Omega_k + \Omega_k\omega = 0, \tag{9}
$$

we may drop this condition $(d^k = 0,\ [k]_q = 0)$ by replacing it with a physical current J on the RHS thus

$$
\begin{aligned}
&D\Omega_k = J_k, \quad J|_{d^k=0,\ [k]_q=0} = 0,\\
&D\Omega^*_{n-k} = \tilde{J}_k, \quad \tilde{J}|_{d^k=0,\ [k]_q=0} = 0,\\
&* : \Lambda^k_q \to \Lambda^{n-k}_q, \quad (f+g)^* = f^* + g^*, \qquad ** : \Lambda^k_q \to \Lambda^k_q,
\end{aligned}
\tag{10}
$$

where $*$ is meant to be an analog of the Hodge dual. When $\tilde{J} = J$ we may impose the duality condition $\Omega^*_{n-k} = \Omega_k$ which may be compared with an instanton solution.

One may define $J := (d\Omega'_k - \omega'\Omega'_k + \Omega'_k\omega')\alpha$, where α is an arbitrary complex function $\alpha : \mathcal{M} \to \mathbb{C}$ and ω' is a known connection defined (supported and/or generated) by a field in some region of $\mathcal{M}$ [4]. That is, the domain of ω contains, and is bigger than that of ω' meanwhile ω' acts as the source of ω. ω in turn may be seen to be simply an analytic continuation (or extension) of ω' via the equation $D\Omega_k = J_k$. This means that outside the support of ω' we must have $d\Omega_k - \omega\Omega_k + \Omega_k\omega = 0$ [4]. In this dynamical picture, the condition $J|_{d^k=0,\ [k]_q=0} = 0$ is no longer necessary and may be completely eliminated. One may not be able to find solutions that satisfy these equations simultaneously for all k and if that happens to be the case, then one may choose the equation(s) that best suit(s) a particular purpose.

Remarks: The equations (10) are similar in form to those of electromagnetism and may therefore be interpreted analogously. The q parameters and $d^k \neq 0$ conditions have each acquired physical significance through the introduction of the supposedly physical currents $J, \tilde{J}$.

IV. ILLUSTRATING Q-SYMMETRIZATION

We wish to illustrate the index structure of q. Consider the q-deformed 2-form f of the gauge potential A which may now be written as

$$
f = dA + qA^2 = (\partial_\mu A_\nu + q_{\mu\nu} A_\mu A_\nu) dx^\mu dx^\nu \equiv f_{\mu\nu} dx^\mu dx^\nu, \tag{11}
$$

where dx^μ are basis 1-forms such that $dx^\mu dx^\nu = q_{\nu\mu} dx^\nu dx^\mu$, $q_{\nu\mu} = \frac{1}{q_{\mu\nu}}$. Then the appropriately symmetrized component form of f is

$$
\begin{aligned}
F_{\mu\nu} &= \frac{1}{2}(f_{\mu\nu} + q_{\mu\nu} f_{\nu\mu}) = \frac{1}{2}\{\partial_\mu A_\nu + q_{\mu\nu} A_\mu A_\nu + q_{\mu\nu}(\partial_\nu A_\mu + q_{\nu\mu} A_\nu A_\mu)\}\\
&= \frac{1}{2}\{\partial_\mu A_\nu + q_{\mu\nu}\partial_\nu A_\mu + q_{\mu\nu}(A_\mu A_\nu + q_{\nu\mu} A_\nu A_\mu)\}.\\
F_{\mu\nu} &= q_{\mu\nu} F_{\nu\mu}.
\end{aligned}
\tag{12}
$$

For electromagnetism (ie. an Abelian case), $A_\mu A_\nu = A_\nu A_\mu$ and

$$
F_{\mu\mu} = \frac{1 + q_{\mu\mu}}{2}(\partial_\mu A_\mu + q_{\mu\mu} A_\mu A_\mu) = 0 \quad \Rightarrow \quad q_{\mu\mu} = -1 \ \ \forall \mu. \tag{13}
$$

Therefore,

$$
\begin{aligned}
F_{\mu\nu} &= \frac{1}{2}\{\partial_\mu A_\nu + q_{\mu\nu}\partial_\nu A_\mu + q_{\mu\nu}(1 + q_{\nu\mu}) A_\mu A_\nu)\}.\\
F^{\mu\nu} &:= \frac{1}{2}\{\partial^\mu A^\nu + q_{\nu\mu}\partial^\nu A^\mu + q_{\nu\mu}(1 + q_{\mu\nu}) A^\mu A^\nu)\}.\\
F^{\mu\nu} &= q_{\nu\mu} F^{\nu\mu}
\end{aligned}
\tag{14}
$$

and the Lagrangian for electrodynamics is

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}\not{D}\psi, \;\; \not{D} = \gamma^{\mu}(\partial_{\mu} + ieA_{\mu}). \tag{15}$$

In $D = 1 + d, \;\; d \geq 2$ the Fourier space propagator

$$\Delta_{\mu\nu}(k) = (k^2\eta_{\mu\nu} + q_{\mu\nu}\; k_{\mu}k_{\nu})^{-1}, \tag{16}$$

will be well defined if the q's are chosen accordingly.

A. The q-symmetrization in general

Let

$$\begin{aligned} &dx^{i_1 i_2 ... i_m} = dx^{i_1}dx^{i_2}...dx^{i_m}, \;\; dx^{i_1 i_2 .. i_{k-1} i_k .. i_m} = q_{i_k i_{k-1}} dx^{i_1 i_2 .. i_k i_{k-1} .. i_m}, \\ &q_{ji} = \frac{1}{q_{ij}}, \end{aligned} \tag{17}$$

then

$$dx^{ijk} = q_{ki}q_{kj}dx^{kij} = q_{ki}q_{ji}dx^{jki} = q_{ji}dx^{jik} = q_{ji}q_{kj}q_{ki}dx^{kji} = q_{kj}dx^{ikj}, \tag{18}$$

and therefore the symmetrized component form of

$$f = f_{ijk}dx^{ijk} \tag{19}$$

is

$$\begin{aligned} F_{ijk} = f^q_{ijk} &= \frac{1}{3!}\{f_{ijk} + q_{jk}q_{ik}f_{kij} + q_{ij}q_{ik}f_{jki} + q_{ij}f_{jik} + q_{ij}q_{ik}q_{jk}f_{kji} + q_{jk}f_{ikj}\} \\ &\equiv \frac{1}{3!}\sum_{p\in S_3} Q_{p;123}\; f_{i_{p(1)}i_{p(2)}i_{p(3)}}, \\ f^q_{ijk} &= q_{ij}f^q_{jik} \end{aligned} \tag{20}$$

In general

$$\begin{aligned} &f = f_{i_1 i_2 ... i_m}dx^{i_1 i_2 ... i_m}, \\ &F_{i_1 i_2 ... i_m} = f^q_{i_1 i_2 ... i_m} = \frac{1}{m!}\sum_{p\in S_m} Q_{p;12...m}\; f_{i_{p(1)}i_{p(2)}...i_{p(m)}}, \end{aligned} \tag{21}$$

where the rule for obtaining the Q's is to first move the rightmost element, of the identity permutation, in the given permutation to its usual (rightmost) position in the identity permutation, while collecting any resulting q-factors with appropriate indices, and then the next rightmost, and next until the identity order of the permutation is restored. When trying to generate the Q's corresponding to the permutation group S_m, it is useful to remember that elements of the mth order permutation group S_m can be obtained from those of S_{m-1}, *as a subset of* S_m, simply by a "cycling" operation C_m. That is,

$$\begin{aligned} S_m &= (1 \oplus C_m \oplus {C_m}^2 \oplus ... \oplus {C_m}^{m-1})S_{m-1} \\ &\equiv S_{m-1} \cup C_m S_{m-1} \cup {C_m}^2 S_{m-1} \cup ... \cup {C_m}^{m-1}S_{m-1}, \\ C_m &: S_m \to S_m, \;\; i_1 i_2 ... i_m \mapsto i_m i_1 i_2 ... i_{m-1}, \quad {C_m}^m = 1. \end{aligned} \tag{22}$$

Thus obtaining the Q's for S_m is easier if the Q's are known for $S_{m-1} \subset S_m$.

V. CONCLUSION

We have proposed a way to directly generate higher curvatures by replacing the $d^N = 0,\ q^N = 1$ condition with a physical current. This has been motivated by the equivalence of the condition $d^N = 0,\ q^N = 1$ to Bianchi's 2nd identity and also by the similarity between the proposed equations and Maxwell's equations for electromagnetism.

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